Global Stabilization of A Class of Switched Cascade Nonlinear Systems

CHEN Xinwei1,2, DIMITROVSKI M. Georgi3,4, MA Ruicheng5, ZHAO Jun1,2
1. State Key Laboratory of Synthetical Automation for Process Industries (Northeastern University), Shenyang, 110819, P. R. China
2. College of Information Science and Engineering, Northeastern University, Shenyang, 110819, P. R. China
3. Faculty of Engineering, Dogus University of Istanbul, Istanbul, Republic of Turkey
4. SS Cyril & Methodius University, School FEEIT, Skopje, Republic of Macedonia
5. School of Mathematics, Liaoning University, Shenyang 110036, P. R. China
E-mail: chen_xinwei@163.com, gdimitrovski@dogus.edu.tr, mrzcheng@163.com, zhaojun@mail.neu.edu.cn

Abstract: In this paper, the problem of the global stabilization for a class of switched cascade nonlinear systems is concerned, whose subsystems are not assumed to be asymptotically stabilizable. Using the convex combination method and the gain-scale factor method, we construct a switching law and design state-feedback controller of individual subsystems to guarantee asymptotic stability of the closed-loop switched nonlinear system. Finally, a simulation example is given to demonstrate the feasibility and effectiveness of the proposed design techniques.

Key Words: Switched cascade nonlinear system, Global stabilization, Convex combination, Gain-scale factor

1 Introduction

Switched systems constitute a special class of hybrid systems which contain both continuous dynamics and discrete dynamics [1]. Switched systems have drawn considerable attention in the last decade due to both theoretical interest and their practical significance (12–16), such as in chemical processes, flight control and network control, etc. Stability is one of the most important issues in the study of switched systems [14]. In general, a switched system does not necessarily inherit the properties of its subsystems. For example, switching between unstable subsystems can give rise to stability, and switching between stable subsystems may lead to instability [17]. For switched systems, many approaches have been put forward to solve (robust) stability and stabilization problems [17–26], such as common Lyapunov function, multiple Lyapunov function, signal Lyapunov function, switched Lyapunov function, average dwell time, and their variants.

On the other hand, the stabilization problem of the switched system is still one challenging problem, which deserves further investigations especially for switched nonlinear systems with all unstable subsystems [27]. Concerning this problem, there are the following two aspects. One is to stabilize a switched system without control input, that is, to stabilize it by designing an appropriate switching law [17]. The other is to stabilize a switched control system via constructing both the controller and switching law [28]. For the stabilization of switched linear system, many approaches have been proposed in [29–31]. [30] investigated the stabilization of planar switched linear control systems and obtained the necessary and sufficient condition to stabilization. [32] obtained that if there exists a stable convex combination of all unstable linear modes, then a periodical switching law can be constructed to make the switched linear system stable. Recently, the stabilization problem for switched nonlinear systems has been studied in [33–35] and the references therein. [36] studied the global stabilization of a class of switched cascade nonlinear systems with linear parts which are switched linear systems under arbitrary switchings. If the linear parts are changed into the nonlinear parts, which will be more interesting and challenging problem.

In this paper, we focus on the global stabilization problem of a class of switched cascade nonlinear systems where all subsystems consist of unstable subsystems via designing both the controller and the switching law. The result of this paper will extend the method formulated in [37, 38] to study the stabilization problem of switched cascade nonlinear systems with the subsystems which can be not stabilization by the method in [37, 38]. Using the convex combination method and the gain-scaling factor method, we construct a switching law and design state-feedback controllers of individual subsystems to guarantee the global stability of such a system at a designed switching law. The results of this paper can also be viewed as a generation of the global stabilization results of non-switched nonlinear systems in [37] in the one switched nonlinear systems. Finally, an numerical example is provided to illustrate the effectiveness of the proposed control method. The main contribution is the co-design of stabilizers and a switching policy proposed to achieve its global stabilization of the switched nonlinear systems.

The rest of the paper is organized as follows. In Section 2, the system description and problem formulation are given. Section 3 presents the main result in this paper. A numerical example is given to show the effectiveness of the proposed scheme in Section 4. Finally, some conclusions are drawn in Section 5.

Notations: The notations throughout this paper are standard. \( R^n \) denotes the set of \( n \)-dimensional real vectors. \( R^{n \times m} \) denotes the set of \( n \times m \)-dimensional real matrices.

For a matrix \( P \), \( P > 0 \) means that \( P \) is a real symmetric and
positive definite matrix. \( \lambda_{\text{max}}(P) \) and \( \lambda_{\text{min}}(P) \) denote the maximum and minimum eigenvalues of \( P \). \( l \) and \( 0 \) denote the identity matrix and zero matrix with appropriate dimensions, respectively. \( \| x \| \) denotes the Euclidean norm of vector \( x \). \( \| A \| \) denotes the induced Euclidean norm of matrix \( A \). \( \| A \|_1 \) denotes the induced 1-norm of matrix \( A \). \text{argmin}[S] denotes the index of minimum element of ordered set \( S \).

## 2 System Description and Problem Formulation

In this paper, we consider a class of switched nonlinear systems that, after a suitable change of coordinates, can be expressed in the following form:

\[
\begin{align*}
\dot{x} &= f_\sigma(t)(x, x), \\
\dot{x} &= A_\sigma(t)x + b_\sigma(t) + g_\sigma(t)(x, u_\sigma(t)),
\end{align*}
\]

where \( x \in \mathbb{R}^n \), \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) is the state vector of the system; the index function \( \sigma(t) : [0, \infty) \rightarrow M = \{1, 2, \ldots, m\} \) is the switching signal which is assumed to be a piecewise continuous (from the right) function depending on time or state or both. \( m \) is the number of models (called subsystems) of the switched system. \( u_k \in \mathbb{R}, \forall k \in M, \) is the control input of the \( k \)-th subsystem. \( f_k(x, t) \) and \( g_k(t, x, u) \) are smooth functions with \( f_k(0, t, 0) = 0 \) and \( g_k(0, t, u_k) = \) smooth functions with \( f_k(0, 0) \) and \( g_k(t, 0, u_k) = 0 \). \( f_k(t, x, u_k) = f_{g_k}(t, x, u_k) = (g_{f_k}, g_{g_k}, \ldots, g_{n_k})^T, \forall k \in M \).

In addition,

\[
A_k = \begin{bmatrix}
0 & a_{1,2,k} & 0 & \cdots & 0 \\
0 & 0 & a_{2,3,k} & \cdots & 0 \\
0 & 0 & 0 & \cdots & a_{n-1,n,k} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{R}^{n \times n},
\]

where \( a_{i,j,k} \) are zero or non-zero constants, \( b = [0, \ldots, 0, 1]^T \in \mathbb{R}^n \).

We assume that the state of the system (1) does not jump at the switching instants, i.e. the trajectory \( x(t) \) is everywhere continuous. We also assume that \( \sigma(t) \) has finite number of switchings on any finite interval of time.

As usually found in the literatures, this is a standard assumption to exclude the Zeno phenomenon which is unacceptable in practice.

The following definition borrowed from [39] will be used in the sequel.

**Definition 1** Let \( \rho(x) \) and \( \phi(x) \) be continuous functions of their arguments, that is, \( x \in \mathbb{R}^n \). We denote \( \phi(x) = o(\mu(x)) \) if there exists a continuous function \( \mu : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( \| \phi(x) \| \leq \mu(\| x \|) \| \rho(x) \| \) for all \( x \in \mathbb{R}^n \) and \( \mu(\| x \|) = 0 \) as \( \| x \| \rightarrow \infty \).

A direct consequence of Definition 1 is that the following holds uniformly in \( t \):

\[
\lim_{\| x \| \rightarrow \infty} \frac{\| \phi(x) \|}{\| \rho(x) \|} = 0.
\]

Our control objective is to find under what conditions and how can the co-design of stabilizers and a switching law be proposed to solve the global stabilization for the switched nonlinear system (1). In addition, a detailed design technique for both the controller and switching law is provided for the switched nonlinear system (1). Now, we give the full characterizations of the switched system (1) via the following assumptions.

**Assumption 1** There exists a set of positive \( \alpha_k \in (0, 1), \) \( k \in M, \) with \( \sum_{k=1}^m \alpha_k = 1 \) and positive definite and radially unbounded functions \( U(z) \) and \( W(z) \), such that

\[
\frac{\partial U(z)}{\partial z} \sum_{k=1}^m \alpha_k f_k(z, 0) < -W(z).
\]

In addition, because of the smoothness of the functions \( f_k(z, x) \), \( \forall k \in M, \) there exist smooth matrix-valued functions \( \Gamma_k(z, x) \) such that

\[
f_k(z, x) - f_k(z, 0) = \Gamma_k(z, x) x.
\]

Let \( m \sum_{k=1}^m \alpha_k \Gamma_k(z, x) x \leq \Gamma(z, x) x.

**Assumption 2** For each fixed \( x \), the functions \( \Gamma_k(z, x) \) satisfies

\[
\left\| \frac{\partial U}{\partial z} \sum_{k=1}^m \alpha_k \| \Gamma_k(z, x) \| \right\| = o(W(z)), \text{as} \| z \| \rightarrow \infty,
\]

where \( U(z) \) and \( W(z) \) are defined in Assumption 1.

Define

\[
\delta(t, x, u) = \sum_{k=1}^m \alpha_k g_k(t, x, u_k) = \begin{bmatrix}
\delta_1(t, x, u) \\
\delta_2(t, x, u) \\
\vdots \\
\delta_n(t, x, u)
\end{bmatrix}
\]

where \( \alpha_k, k = 1, 2, \ldots, m, \) are defined in Assumption 1.

**Assumption 3** For the function \( \delta(t, x, u) \) in (4), there exists a function \( \lambda(\epsilon) \geq 0 \), such that for \( \epsilon > 0 \),

\[
\sum_{i=1}^n \epsilon_i \cdot |\delta_i(t, x, u)| \leq \lambda(\epsilon) \sum_{i=1}^n \epsilon_i \cdot |x_i|.
\]

**Assumption 4** For matrix \( A = \sum_{i=k}^m \alpha_k A_k \), we assume that

\[
\sum_{i=k}^m \alpha_k a_{i,i+1,k} \neq 0 \text{ for } i = 1, 2, \ldots, n - 1.
\]

**Remark 1** Since \( a_{i,i+1,k} \) is non-zero, each individual subsystem of the system (1) does not satisfy the restrictions on the structure of the nonlinear system studied in [37] where \( a_{i,i+1,k} = 1, i = 1, 2, \ldots, n - 1 \). In this paper, there may be some \( j \) such that \( a_{j,j+1,k} = 0 \). Therefore, the proposed method in [37] is no longer applicable. In fact, a similar assumption in (5) is adopted in [37] for non-switched nonlinear system to study the global stabilization.
Remark 3 If at least one of the subsystems is globally asymptotically stabilizable, this problem of the global stabilization studied in this paper will be trivial. Therefore, none of the individual subsystems is assumed to be asymptotically stable in this paper.

3 Main Result

In this section, we will design state-feedback controllers for subsystems and construct a state-dependent switching signal to globally stabilize the switched nonlinear system (1).

We are now in the position to state our main result.

Theorem 1 Consider the switched nonlinear system (1) satisfying Assumptions 1-4, and \( \gamma(e) \triangleq e^{-1} - 2\sqrt{n} \| P \| L(e) > 0 \). Then, there exists a class of state-feedback controller of the form

\[
u_k(x) = K(e)x
\]

with \( K(e) = \left[ \begin{array}{ccc} \frac{\epsilon}{k}, & \cdots, & \frac{\epsilon}{k} \end{array} \right], \epsilon > 0, \forall k \in M, \) for all subsystems and the switching signal

\[
\sigma(t_0) = \arg\min_{k \in M} \{ \Omega_k | \xi_0 \in \Omega_k \},
\]

\[
\sigma(t) = \begin{cases} k, & \text{if } \sigma(t^-) = k, \\
\arg\min_{k \in M} \{ \Omega_k | \xi(t) \in \Omega_k \}, & \text{if } \sigma(t^-) = k, \\
\sigma(t^-), & \text{if } \xi(t) \notin \Omega_k, \end{cases}
\]

such that the closed-loop switched system (1) under the designed switching law (7) is globally asymptotically stable at the origin, where \( \xi = [z^T, x^T]^T \),

\[
\Omega_k = \left\{ \xi | \begin{array}{c} \frac{\partial U(z)}{\partial z} f_k(z, 0) < -W(z), \\
\| \frac{\partial U(z)}{\partial z} \| \| \Gamma_k(z, x) \| = o(W(z)), \\
\| \frac{\partial U(z)}{\partial z} [A_k(e)x + g_k] \| \leq -\gamma(e) \| E(e)x^2 \| \end{array} \right\}
\]

with \( \gamma(e) \triangleq e^{-1} - 2\sqrt{n} \| P \| L(e) > 0 \) and \( A_k(e) = A + bk(e) \).

Proof: Substituting the designed controller (6) into the switched system (1), one has

\[
\dot{z} = f_{\sigma(t)}(z, x),
\]

\[
\dot{x} = \begin{cases} b_n f_{\sigma(t)}(t, x, u_{\sigma(t)}(x)), & \text{if } \sigma(t) = k, \\
\end{cases}
\]

With the help of the Assumption 1 and (4) and by convex combination technique, we can construct a non-switched nonlinear system as follows:

\[
\dot{z} = \sum_{i=1}^{n} \alpha_k f_k(z, x), \\
\dot{x} = A_c(e)x + \delta(t, x, u).
\]

Define \( E(e) = \text{diag} \{ 1, e, \cdots, e^{n-1}, e^n \} \). Then, similar to [37], one can easy to verify that

\[
E(e)^{-1}A_c(e)E(e) = A_c(e).
\]

By using the fact that \( A_cP + PA_c^T = -I \) in Remark 1, we choose a Lyapunov function

\[
V(x) = x^T P(e)x,
\]

where \( P(e) = E(e)P e(e) \).

One can easily compute that

\[
A_c(e)P(e) + P(e)A_c(e)^T = -e^{-1}E(e)^2.
\]

Now, we first consider the following nonlinear system in (10)

\[
\dot{x} = A_c(e)x + \delta(t, x, u).
\]

Considering the system (12), the time-derivative of \( V(x) \) along the trajectory of the system (12) satisfies that

\[
\dot{V} = \frac{\partial V}{\partial x} \dot{x} = \frac{\partial V}{\partial x} (A_c(e)x + \delta(t, x, u))
\]

\[
= -e^{-1}x^T E(e)x + 2x^T P(e)\delta(t, x, u)
\]

\[
= -e^{-1} \| E(e)x^2 \| + 2x^T P \delta(t, x, u). \quad (13)
\]

Using Assumption 3, we obtain that

\[
\| E(e)\delta(t, x, u) \| \leq L(e) \| E(e)x \| \leq \sqrt{n}L(e) \| E(e)x \| \quad (14).
\]

Substituting (14) into (13), one has

\[
\dot{V} = \frac{\partial V}{\partial x} \dot{x} \leq -e^{-1} - 2\sqrt{n} \| P \| L(e) \| E(e)x \|^2. \quad (15)
\]

Since \( \gamma(e) = e^{-1} - 2\sqrt{n} \| P \| L(e) > 0 \), then the system (12) is globally asymptotically stable by (15).

On the other hand, with the help of Assumption 1 and 2, we have

\[
\dot{U}(z) = \frac{\partial U}{\partial z} \sum_{i=1}^{n} \alpha_k f_k(z, 0)
\]

\[
+ \frac{\partial U}{\partial z} \left( \sum_{i=1}^{n} \alpha_k (f_k(z, x) - f_k(z, 0)) \right)
\]

\[
\leq -W(z) + \frac{\partial U}{\partial z} \| \Gamma_k(z, x) \| \leq -W(z) + \frac{\partial U}{\partial z} \| \Gamma_k(z, x) \| \| k(16) \|
\]

In view of (15), (16) and Assumption 2, using the similar method as Theorem 3 in [39], we can obtain that the system (10) is globally asymptotically stable at the origin.

From the switching signal (7), we know that, \( \forall \xi(t) \in R^n \), there exists a \( k \in M \), such that for any \( \xi(t) \in \Omega_k \),

\[
\frac{\partial U(z)}{\partial z} f_k(z, 0) < -W(z),
\]

\[
\left\| \frac{\partial U(z)}{\partial z} \right\| \| \Gamma_k(z, x) \| = o(W(z)),
\]

\[
\left\| \frac{\partial U(z)}{\partial z} \right\| \left( A_k + bK(e) \right) x + g_k \leq -\gamma(e) \| E(e)x^2 \| \quad (17)
\]

For the definition of switching law (7), we have that \( \bigcup_{k=1}^{n} \Omega_k = R^n \setminus \{ 0 \} \). Therefore, global stabilization of the closed-loop switched system (1) with (6) is achieved under the switching signal (7).

Remark 4 For the convex combination system, if the zero dynamics in the switched system (1) meet the input-to-state stability (ISS) with respect to the input \( x \), then Assumption 2 is trivial.
4 An Illustrative Example

In this section, a numerical example will be studied to demonstrate the effectiveness of the proposed design method by co-designing individual controllers and a switching signal to guarantee global stabilization.

Consider the switched nonlinear system (1) consisting of two subsystems

\[
\begin{align*}
\dot{z} &= f_k(z, x), \\
\dot{x} &= \bar{A}_k x + bu_k + \bar{g}_k(t, u_k), \\
k &= 1, 2,
\end{align*}
\]

where \( f_1(z, x) = -z + x^2, f_2(z, x) = -2x^3 + x^2, \bar{A}_1 =
\begin{bmatrix}
1 & 0 & 0 \\
-3 & -1 & 2 \\
0 & 1 & 0
\end{bmatrix},
\bar{A}_2 =
\begin{bmatrix}
1 & 2 & 0 \\
-3 & -1 & 0 \\
0 & 1 & 0
\end{bmatrix},
\begin{bmatrix}
b \\
10x_2 \sin(2t) \\
0
\end{bmatrix}.
\]

Such a switched system can be written as the form of (1) if

\[
A_1 =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{bmatrix},
\bar{g}_1 =
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\begin{bmatrix}
x_1 + x_1 \sin(2t) \\
10x_2 \sin(2t) - 3x_1 - x_2
\end{bmatrix}.
\]

\[
A_2 =
\begin{bmatrix}
0 & 2 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\bar{g}_2 =
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\]

It is easy to verify that both subsystems are not global stabilizable based on the existing methods in [37]. However, the convex combination system of (18) satisfies the conditions of Theorem 1, where \( a_1 = a_2 = \frac{1}{2} \). Then, with the method in Remark 1, one can compute that

\[
K =
\begin{bmatrix}
-3 & -4 & -5
\end{bmatrix},
P =
\begin{bmatrix}
8.5 & 2.7 & 0.3 \\
2.7 & 2.2 & 1/3 \\
0.3 & 1/3 & 1/6
\end{bmatrix}.
\]

Define \( V_0(z) = \frac{1}{2}z^2 \). Then, for \( z \)-dynamics of the obtained convex combination system, one has

\[
\dot{V}_0(z) \leq -z^4, \forall |z| \geq x_1^2.
\]

Therefore, the globally asymptotically stable of the closed-loop switched nonlinear system under the switching signal (7) follows from Theorem 1, where the controllers of the subsystems

\[
u_k =
\begin{bmatrix}
-\frac{3}{e^2} & -\frac{4}{e^2} & -\frac{5}{e^2}
\end{bmatrix} x
\]

with \( \epsilon = 0.1 \).

With the help of the designed controller (20) and switching signal (7), the simulation is carried out with the initial state \([0.5, 0.1, -0.1, 0.01]^T\). The simulation results are depicted in Figs. 1 and 2.

Figure 1 shows the state response of the closed-loop switched system (18) under the designed switching signal (7), which indicates that the closed-loop switched system is asymptotically stable. Figure 2 gives the switching signal. Thus, the simulation results well illustrate the effectiveness of the proposed method.

5 Conclusions

This paper has investigated the global stabilization problem of a class of switched cascade nonlinear systems, whose subsystems are not assumed to be asymptotically stabilizable. The sufficient conditions for stabilization have obtained. Both the controller for the individual subsystem and the switching law have been constructed by using the convex combination method and gain-scale factor for global stabilization of switched nonlinear systems. The final simulation example of the effectiveness of the method. Finally, the effectiveness of the proposed method has been demonstrated through a numerical example. Moreover, the result can be regarded as an extension of the stabilizing results for non-switched nonlinear systems in [37].

References