Filter-Based $H_{\infty}$ Output Tracking Control for Switched Discrete-Time Linear Systems in the Absence of State Measurements

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Abstract: This paper deals with the filter-based $H_{\infty}$ output tracking and model following control problem for a class of switched discrete-time linear systems. We don’t suppose the entire knowledge of the system state to design switching laws and design tracking controllers. We only use the measured output tracking error to design the switching law, which together with the designed filter-based tracking controllers can guarantee the $H_{\infty}$ output tracking problem of the closed-loop switched system is solvable. The method is proposed to ensure in the same time the $H_{\infty}$ tracking and the filtering processes.

Key Words: Switched discrete-time systems, multiple Lyapunov functions, switching law, filter-based output tracking control

1 Introduction

Tracking control, which is split into state tracking and output tracking, is an important problem in control field and has wide applications in dynamic processes in economics, biology and other practical fields. The main objective of output tracking control is trying to minimize the error between the output of the plant and the output of a given reference model by a designed tracking controller [1-3]. The issue of output tracking control has been well addressed for non-switched systems [4-8]. Robust $H_{\infty}$ output tracking problem was considered in [3] for linear time delay systems, and sufficient conditions were established to guarantee the existence of linear robust tracking controller. A less conservative result of $H_{\infty}$ output tracking control for neutral systems was investigated in [2].

On the other hand, in recent years, switched systems have attracted considerable attention due to theoretical importance and practical applications. The motivation of such systems also arise from the better performance achieved via imposing a controller switching strategy [9, 10]. Stability analysis and control synthesis are two key problems in the study of switched systems. There are many methods are used in the study of switched systems, such as, finding a common Lyapunov function for all the subsystems and solving the problems under an arbitrary switching law [9, 11]. However, most switched systems in practice do not possess a common Lyapunov function or is very hard to find. Thus, adoption of the multiple Lyapunov function method is more practical to overcome the shortcoming, and it solves the problems for switched systems under certain designed switching law [12, 13]. There are some other approaches [14-16].

Since the development of switched control in intelligent robot and its applications in industries, the study of tracking control of switched systems have become more important. For switched continuous-time systems, sufficient conditions for the solvability of the state tracking control problem were given in [17] via the state-dependent switching method. Yet, the system state information is often unavailable in many control systems and applications. The problem of observer-based state tracking control was further considered in [18] for a class of switched linear systems by utilizing single Lyapunov-Krasovskii functional method. But, a convex combination condition need to be satisfied in [18]. In fact, the convex combination may not exist or is usually difficult to find its coefficients. In [19], exponential $H_{\infty}$ output tracking control for switched neutral system was investigated via average dwell time method. A method for designing switching rules that enforce the state of the switched dynamic system to track a given reference signal was proposed in [20]. However, for switched discrete-time systems, the results on the tracking problem are very limited. [21] considered the exponential $L_{2} - L_{\infty}$ output tracking control problem for switched discrete-time linear systems using the average dwell time approach. However, two basic assumptions in [21] are that the full state is measurable and the tracking control problem for each individual subsystem is solvable. But they are not common cases in practice. [22] studied the $H_{\infty}$ output tracking control for switched discrete-time systems via static output feedback. These motivated the present study.

In this paper, we study the filter-based $H_{\infty}$ output tracking control problem for switched discrete-time systems. Compared with the existing results on the tracking control for switched systems, this paper owns three distinct features.

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First of all, we consider the case that the full state information is unavailable and therefore avoid using it. Instead, we design an output error-dependent switching law and design filter-based tracking controllers.

Second, we don’t require that the system matrix of the reference model is Hurwitz, while in the existing results, the system matrix of the reference model is required to be Hurwitz.

Finally, we don’t need the solvability of the output tracking problem for each individual subsystem. Besides, the condition is developed to ensure in the same time the tracking and the filtering processes.

2 Problem statement and preliminaries

In this section, we will describe the filter-based tracking controller implemented. Therefore, we consider the following problem concerning in this paper is to design such a switching law for the switching signal; \( u_r(k) \) is generated by the reference signal; \( A_i, B_i^1, B_i^2, C_i \) and \( D_i^1, i \in M \) are system matrices with compatible dimensions.

The reference output \( y_r(k) \) is generated by the reference model

\[
\begin{align*}
\dot{x}(k+1) &= A_i x(k) + B_i u_i(k), \\
y(k) &= C_i x(k) + D_i u_i(k),
\end{align*}
\]

where \( x(k) \in R^n \) is the state vector, \( u(k) \in R^m \) is the input vector, \( y(k) \in R^p \) is the measured output vector, \( o(k) \in R^q \) is the exogenous disturbance signal which is assumed to belong to \( l_2[0, \infty) \); \( \sigma(k) : [0, \infty) \to M = \{1, 2, \ldots, N\} \) is the switching signal; \( A_i, B_i^1, B_i^2, C_i \) and \( D_i^1, i \in M \) are system matrices with compatible dimensions.

The problem concerning in this paper is to design a switching law and tracking controllers such that the system output \( y(k) \) can track the reference output \( y_r(k) \).

Remark 1: For model reference tracking control problem, the system matrix of the reference model is required to be Hurwitz in the existing results [17-19, 21], while in this paper, the system matrix \( A_i \) of the reference model is not required to be Hurwitz.

In practice, the whole state is often unavailable from measurements and the control approach via state feedback and state-dependent switching law would not be able to be implemented. Therefore, we consider the following filter-based tracking controller

\[
\begin{align*}
\dot{x}(k+1) &= A_i x(k) + B_i^1 u_i(k) + B_i^2 o(k), \\
y(k) &= C_i x(k) + D_i u_i(k),
\end{align*}
\]

where,

\[
u(k) = u_r(k) + u_e(k),
\]

(3)

\[
(4)
\]

\[
\begin{align*}
u_r(k) &= -K_{i} \dot{x}(k), \\
u_e(k) &= (K_{2} + K_{1}) y_{r}(k) + Q_{1} o_{r}(k) ,
\end{align*}
\]

(5)

\[
\begin{align*}
\dot{x}(k+1) &= A_{f} x(k) + B_{f} o(k) + C_{f} (y(k) - D_{f} o(k)),
\end{align*}
\]

\[
\begin{align*}
u(k) &= u_r(k) + u_e(k),
\end{align*}
\]

(4)

where,

\[
u_r(k) = -K_{i} \dot{x}(k),
\]

\[
u_e(k) = (K_{2} + K_{1}) y_{r}(k) + Q_{1} o_{r}(k) ,
\]

(5)

\[
\begin{align*}
\dot{x}(k+1) &= A_{f} x(k) + B_{f} o(k) + C_{f} (y(k) - D_{f} o(k)),
\end{align*}
\]

(4)

Our objective is to design the tracking controller in the form of (4) and design a switching law \( \sigma(e(k)) \) such that the filtering error \( e_r(k) = x(k) - \dot{x}(k) \) and the output tracking error \( e(k) = y(k) - y_{r}(k) \) satisfy

\[
\begin{align*}
\lim e_r(k) = 0 \\
\lim e(k) = 0,
\end{align*}
\]

(2)

Under the zero-initial condition,

\[
\|e(k)\|_2 \leq \gamma \|o(k)\|_2
\]

holds for all nonzero \( o(k) \in l_2[0, \infty) \), where \( \gamma > 0 \) is a constant number.

3 Main results

In this section, we will give sufficient condition for the solvability of the \( H_{\infty} \) output tracking problem of the switched discrete-time system (1) based on the filter-based tracking controller (4) and the switching law \( \sigma(e(k)) \).

Theorem 1. Consider the \( H_{\infty} \) output tracking control problem of the system (1). Given scalars \( \beta_i \geq 0 \) and \( \gamma > 0 \). If

(a) There exist matrices \( K_{2i}, G \) and \( Q_i \) that satisfy the following matrix equations:

\[
\begin{align*}
A_i G - G A_i + B_i^1 K_{2i} &= 0, \\
B_i^1 Q_i - G B_i &= 0, \\
D_i^1 Q_i - D_i &= 0, \\
C_i G - C_i D_i K_{2i} &= 0.
\end{align*}
\]

(6)

(b) There exist a set of positive definite matrices \( \hat{P}_j \) and matrices \( A_j, K_j \) satisfying

\[
\begin{align*}
\begin{bmatrix}
\hat{A}_j \hat{P}_j \hat{A}_j - \hat{P}_j + \sum_{j \neq j} \beta_j (\hat{P}_j - \hat{P}_j) \\
(\hat{A}_j \hat{P}_j B_j^2)^T \\
\hat{C}_j
\end{bmatrix} &< 0.
\end{align*}
\]

(7)

then, under the switching law

\[
\sigma(e(k)) = \arg \min_{i \in M} \{e^T(k) W_i e(k)\},
\]

(8)
the $H_{\infty}$ output tracking control problem of the system (1) is solved, where, \( \tilde{P}_i = \begin{bmatrix} P & 0 \\ 0 & W_i \end{bmatrix} \) and $C_{\beta}$ are obtained by $A_i - A_{\beta_i} - C_{\beta} C_i = 0$.

**Proof.** Introducing the following transformations
\[
\tilde{x}(k) = (k) - G x_{\infty}(k) , \ e(k) = y(k) - y_{\infty}(k) , \ e_{\beta}(k) = x(k) - \tilde{x}(k) .
\]

In terms of the transformations (9), the filter (3) and the controller (4), and using (6) one has the resulting closed-loop switched system
\[
\begin{aligned}
\dot{\tilde{x}}(k+1) &= \tilde{A}_{\sigma(k)} \tilde{x}(k) + \tilde{B}_{\sigma(k)} e(k), \\
\dot{e}(k) &= \tilde{C}_{\sigma(k)} \tilde{x}(k),
\end{aligned}
\]
where,
\[
\tilde{x}(k) = \begin{bmatrix} T(k) \\ e_\beta(k) \end{bmatrix} , \quad \tilde{A} = \tilde{A}_i + \tilde{B}_i \tilde{K}_i , \quad \tilde{C}_i = \tilde{C}_i + \tilde{D}_i \tilde{K}_i , \\
\tilde{B} = \begin{bmatrix} B_i^T \\ B_i \end{bmatrix} , \quad \tilde{K}_i = \begin{bmatrix} -K_{ii} & 0 \\ 0 & -\tilde{A}_i \end{bmatrix} , \quad i \in M .
\]

Choose the Lyapunov function candidate for the system (10) as
\[
V(\tilde{x}(k)) = V_{\sigma(k)}(\tilde{x}(k)) = \min_{i \in \mathcal{M}} \left\{ V_i(\tilde{x}(k)) \right\} ,
\]
where
\[
V_i(\tilde{x}(k)) = \tilde{x}_i^T(k) \tilde{P}_i \tilde{x}(k) = \begin{bmatrix} T(k) \\ e_\beta(k) \end{bmatrix} \begin{bmatrix} P & 0 \\ 0 & W_i \end{bmatrix} \begin{bmatrix} T(k) \\ e_\beta(k) \end{bmatrix} .
\]

We know that $V_i(\tilde{x}(k))$ can be written as $V_i(\tilde{x}(k)) = \tilde{x}_i^T(k) \tilde{P}_i \tilde{x}(k) + \min_{i \in \mathcal{M}} \left\{ e_\beta^T(k) W_i e(k) \right\}$.

Now, we divide the proof into two cases.

**Case 1:** When $\sigma(k+1) = \sigma(k) = i$ , in view of (7) and switching law (8), we obtain
\[
\Delta V(\tilde{x}(k)) + e_\beta^T(k) e(k) - \gamma^2 \omega_\beta^T(k) \omega(k) = V_i(\tilde{x}(k+1)) - V_i(\tilde{x}(k)) + e_\beta^T(k) e(k) - \gamma^2 \omega_\beta^T(k) \omega(k) < 0 .
\]

**Case 2:** Without loss of generality, when $\sigma(k+1) = j \neq \sigma(k) = i$ , from switching law (8) we have
\[
\Delta V(\tilde{x}(k)) + e_\beta^T(k) e(k) - \gamma^2 \omega_\beta^T(k) \omega(k) = V_j(\tilde{x}(k+1)) - V_j(\tilde{x}(k)) + e_\beta^T(k) e(k) - \gamma^2 \omega_\beta^T(k) \omega(k) \leq V_j(\tilde{x}(k+1)) - V_j(\tilde{x}(k)) + e_\beta^T(k) e(k) - \gamma^2 \omega_\beta^T(k) \omega(k) < 0 .
\]

Then, by considering $V(\tilde{x}(k))$ as the overall Lyapunov function of the switched system (10) and from (13) and (14), we know that $\Delta V(\tilde{x}(k)) < -e_\beta^T(k) e(k) + \gamma^2 \omega_\beta^T(k) \omega(k)$.

Therefore, the inequality $\| e(k) \|_2 < \gamma \| \omega(k) \|_2$ holds for all nonzero $\omega(k) \in L_2[0, \infty)$.

From $\omega(k) = 0 \in L_2$, we notice that (13) and (14) imply asymptotic stability of the switched system (10) and $\lim_{k \to \infty} e(k) = 0$. This completes the proof.

**Remark 2:** Introducing the item \( \sum_{j=1}^{N} \beta_j (\tilde{P}_j - \tilde{P}_i) \) in Theorems 1 allows the $H_{\infty}$ output tracking problem not to be solvable for each subsystem, but this problem is still solvable for switched systems.

Since the inequalities in (7) are bilinear in $\tilde{K}_i$ and $\tilde{P}_i$, the design problem can’t be solved by using the Matlab LMI Control Toolbox, which may lead to computational difficulty. Therefore, we will present LMI condition which has certain conservatism to solve the output tracking problem in the next theorem.

**Theorem 2.** Given scalars $\beta_{\beta} > 0, \gamma > 0$ and $\lambda_i$. If the condition (a) of the Theorem 1 is satisfied and if there exist symmetric matrices $X_i > 0, \ X_j > 0$, matrices $R_i$ and $Y_i$ for all $i, j = 1, 2, \cdots, m, i \neq j$ such that the set of LMIs
\[
\begin{bmatrix}
(X_i - R_i - R_i^T)(1 + \sum_{j=1}^{N} \beta_j) & 0 \\
\tilde{A}_i, & (Y_i^T + (B_i^T Y_i)^T)
\end{bmatrix} < 0 ,
\]
hold, then, under the switching law
\[
\sigma(e(k)) = \arg \min_{i \in \mathcal{M}} \{ e_\beta^T(k) W_i e(k) \} = \arg \min_{i \in \mathcal{M}} \{ V_i(\tilde{x}(k)) \} ,
\]
the $H_{\infty}$ output tracking control problem of the system (1) is solved, where, \( X_i = \tilde{P}_i^{-1} = \begin{bmatrix} P_i^{-1} & 0 \\ 0 & W_i^{-1} \end{bmatrix} \), \( R_i = \begin{bmatrix} R_{i11} & 0 \\ 0 & \lambda_i R_{i11} \end{bmatrix} \), \( Y_i = \begin{bmatrix} -U_i & \lambda_i U_i \\ 0 & \lambda_i V_i \end{bmatrix} \).

Once the LMI condition of Theorem 2 has feasible solutions, the associated controller gain matrices are readily derived from $K_{ii} = U_i R_{i11}^{-1}$, $A_{\beta_i} = V_i R_{i11}^{-1}$ and $C_{\beta}$ are obtained by $A_i - A_{\beta_i} - C_{\beta} C_i = 0, \forall i \in M$.

**Proof.** Since $X_i > 0$ and $X_i \leq R_i + R_i^T$ by (15), it turns out that $R_i$ is nonsingular, and it is easy to see that $-R_i^T X_i^{-1} R_i \leq X_i - R_i - R_i^T$. Hence (15) implies
\[
\begin{bmatrix}
(-R_i^T X_i^{-1} R_i)(1 + \sum_{j=1}^{n} \beta_j) & 0 & (\overline{A}_i R_j)^T + (B_i^T Y_i)^T \\
0 & -\gamma^2 I & \overline{B}_i^T \\
\overline{A}_i R_j + B_i^T Y_i & \overline{B}_i^T & -X_j \\
\overline{C}_i R_j + \overline{D}_j Y_i & 0 & 0 \\
\sqrt{\beta_i} R_i & 0 & 0 \\
\vdots & \vdots & \vdots \\
\sqrt{\beta_m} R_i & 0 & 0 \\
(\overline{C}_i R_j)^T + (\overline{D}_j Y_i)^T & \sqrt{\beta_i} R_i^T & \cdots & \sqrt{\beta_m} R_i^T \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & 0 \\
0 & -X_i & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & -X_m \\
\end{bmatrix} < 0, \quad (17)
\]

By multiplying (17) from the left by diag \{\gamma_i^2 I, I, I, \ldots, I\} and from the right by diag \{\gamma_i^2 I, I, I, \ldots, I\}, and by a Schur complement, we obtain
\[
\begin{bmatrix}
-X_i^{-1} + \sum_{j=1}^{n} \beta_j (X_j^{-1} - X_i^{-1}) & 0 & \overline{A}_i^T & \bar{C}_i \\
0 & -\gamma^2 I & \overline{B}_i^T & 0 \\
\overline{A}_j & \overline{B}_j & -X_j & 0 \\
\bar{C}_j & 0 & 0 & -I \\
\end{bmatrix} < 0, \quad (18)
\]

which is equivalent to (7) by the Schur complement, where \(X_i^{-1} = \bar{P}_i\). The result then follows by Theorem 1.

**Remark 3:** Given the above established result, we know that a natural problem is to minimise the upper bound of the disturbance attenuation for the system (10). Thus, the scalar \(\gamma\) can be optimized by the following optimization problem.

\[
\text{Min } \delta \text{ subject to LMI (15), } i, j \in M_i, \ i \neq j \text{ with } \delta = \gamma^2 \text{ over } X_i, X_j, R_i, Y_i.
\]

(19)

The minimum disturbance attenuation level bound is then obtained by \(\gamma = \sqrt{\delta^*}\), where \(\delta^*\) is the optimal value of \(\delta\).

Once this optimization problem is solved, and the corresponding controller gain matrices are obtained.

### 4 Conclusions

In this paper, we have studied the filter-based \(H_\infty\) output tracking and model following control problem for a class of switched discrete-time linear systems using the multiple Lyapunov functions method. Sufficient condition has been developed to guarantee that the output of the switched system tracks the output of the given reference model in the \(H_\infty\) sense by utilizing the tracking controllers and the switching technique. A switching law based on the output tracking error and filter-based tracking controllers for subsystems, respectively, have been designed when the whole system state is not measurable.

### References


