Tracking of Output-constrained Switched Nonlinear Systems in Strict-feedback Form

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Abstract: This paper is concerned with the tracking for a class of output-constrained switched nonlinear system in strict-feedback form. A Barrier Lyapunov Function, which grows to infinity when its arguments approach some limits, is introduced to ensure that the state constraint is not violated at any time. Bounded state feedback controllers of individual subsystems and a common Lyapunov function (CLF) are explicitly constructed to ensure that output of the closed-loop system track a given constant signal under arbitrary switchings. An example is given to show the effectiveness of the proposed method.

Key Words: Switched systems, Output-constrained, Barrier Lyapunov, Backstepping.

1 Introduction

In recent decade, switched systems have been extensively studied by many researchers [1]-[6], to name just a few. Meanwhile, because of explicit engineering background and extensive application prospects, switching control is progressively applied to networks [7], circuit and power systems [8], aeronautics and astronautics [9][10], and so on. When the switching mechanism is either unknown or too complicated to be useful in the stability analysis, find conditions that guarantee asymptotic stability of a switched system for arbitrary switching signals is a significant issue [2]. It is well known that if we can find a CLF for all subsystems, we can obtain stability of the switched system under arbitrary switchings [26]. However, it is more difficult to find a CLF for a switched nonlinear system than for a switched linear system, due to the different coordinate transformation for different subsystem, even if the switched nonlinear system in strict-feedback form.

Backstepping design method is a power tool for designing a stabilizing controller for a non-switched nonlinear system in strict-feedback form [11][12]. This design method is of course expected to be useful for stabilizing a switched nonlinear system. However, backstepping cannot be directly extended to the switched nonlinear system due to the different coordinate transformation for different subsystem according to the standard backstepping procedure. Recently, there are some results on the global stabilization problem for switched nonlinear system in strict-feed form under arbitrary switchings by backstepping [15][17]. Meanwhile, [18] and [27] investigates the global stabilization problem for a class of switched nonlinear systems in p-normal form by a so-called power integrator backstepping design method [13][14]. However, it is worth pointing out that many systems in practical exist states/output constraints. Therefore, the constrained control problem is of great significance. Unfortunately, these constraints are not taken into account in the papers aforementioned.

There exist various techniques to solve the constrained control problem for linear systems [22][23] and nonlinear systems [24][25]. A Barrier Lyapunov Function is proposed in [19] to deal with the domain globally asymptotical stabilization problem for a non-switched feedback linearizable system which can be transformed into a Brunovsky normal form with state constraints. Subsequently, this Barrier Lyapunov Function is developed to solve the tracking problem of a class of non-switched nonlinear systems in strict-feedback form with an output constraint [20]. Nevertheless, to the best of our knowledge, the tracking problem of output-constrained switched nonlinear system in strict-feedback form has no results so far, which motivates the present study.

This paper investigates the tracking problem for a class of output-constrained switched nonlinear systems in strict-feedback form. The tracking under arbitrary switchings without violating the output constraint can be obtained by the bounded state feedback controllers which are explicitly constructed based on the backstepping. Different from the Quadratic Lyapunov Function, the Lyapunov function in this paper consists of two parts: a Barrier Lyapunov Function and a Quadratic Lyapunov Function. The Barrier Lyapunov Function can protect the constrained output from violating the constraint, and the Quadratic Lyapunov Function is chosen for the free-constrained states. It is well-understood that a key point in the backstepping design procedure for a switched nonlinear system is to find a common coordinate transformation at each step. Usually, we can get a CLF only if we can construct a common stabilizing function at each step. In this paper, we explicitly construct the stabilizing function at each step, then we can obtain a CLF at the last step. The result of this paper has two distinct features. Firstly, compared with [16], we study the tracking problem rather than the stabilization problem, and the method to construct the controller is totally different from [16]. Secondly, different from [18][23], this paper investigates the tracking problem and takes the state constraint into account.

This paper is organized as follow: Section 2 introduces a basic definition, basic lemmas, and gives the problem state-
moment. Section 3 shows how to design state feedback controller and a CLF by the backstepping technique. Section 4 gives a numerical example to illustrate the effectiveness of our proposed method and the conclusion is presented in the last section.

2 Preliminaries

2.1 Basic Definition

Firstly, we introduce a basic definition which will be used in this paper.

Definition 1[19]. A Barrier Lyapunov Function is a scalar function $V(x)$, define with respect to the system $\dot{x} = f(x)$ on an open region $\mathcal{D}$ containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of $\mathcal{D}$, has the property $V(x) \to \infty$ as $x$ approaches the boundary of $\mathcal{D}$, and satisfies $V(x(t)) \leq b$, $\forall t \geq 0$ along the solution of $\dot{x} = f(x)$ for $x(0) \in \mathcal{D}$ and some positive constant $b$.

A Barrier Lyapunov Function may be symmetric or asymmetric, as illustrated in Fig.1. In this paper, we employ the symmetric Barrier Lyapunov Function.

![Fig. 1: Schematic diagram of a Symmetric and an Asymmetric Barrier Lyapunov Function.](image)

2.2 Problem Statement

Consider a class of switched nonlinear system of the form:

$$
\begin{align*}
\dot{x}_1 &= g_{1,\sigma(t)}(x_2), \\
\dot{x}_i &= f_{i,\sigma(t)}(x_1) + g_{i,\sigma(t)}(x_i)x_{i+1}, \quad i = 2, \ldots, n-1, \\
\dot{x}_n &= f_{n,\sigma(t)}(x_n) + g_{n,\sigma(t)}(x_n)u_{\sigma(t)}, \\
y &= x_1,
\end{align*}
$$

where $x_1 := (x_1, \ldots, x_i)^T \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$ is the system state; $\sigma(t) : [0, +\infty) \to M = \{1, 2, \ldots, m\}$ is the switching signal; $u_n \in \mathbb{R}$ is the control input of the $k$-th subsystem. All functions are continuously differentiable functions with $f_{i,k}(0) = 0$, $i = 2, \ldots, n$ and $0 < g \leq g_{i,k}(\bar{x}_i) \leq g$, $g, \gamma$ are positive constants, $i = 1, 2, \ldots, n$, $\forall k \in M$.

There exist magnitude constraints on the system output due to physical/performance limits as follows:

$$|y(t)| < y_{lim}, \quad \forall t \geq 0, \quad y_{lim} > 0. \tag{2}$$

Our control objective is to solve the following output tracking problem: Design state feedback controllers to ensure that the output of the system (1) tracks a given constant signal $y_r > 0$ with constraint (2) under arbitrary switching.

Assumption 1. For $i = 2, \ldots, n$,

$$|f_{i,k}(x_i)| \leq (|x_2|^2 + \ldots + |x_i|^2)\mu_{i,k}(\bar{x}_i), \quad \forall k \in M,$$

where $\mu_{i,k}(\bar{x}_i)$ is a set of known non-negative smooth functions.

Remark 1. A similar assumption in study of the tracking problem of non-switched system in [14] can be regarded as a special case of Assumption 1 with $\mu_{i,k} = 1, \forall k \in M$.

2.3 Basic Lemmas

The following lemmas will be used in the backstepping procedure and the proof of Theorem 1. Lemma 1 can be regarded as a direct extension for switched systems from [20], therefore, due to space limitation, we omit the proof of Lemma 1.

Lemma 1. For any positive constants $a, b$, let $\mathbb{Z} := \{z_1| -a < z_1 < b\}$. Consider the switched system:

$$\dot{z} = \phi_{\sigma(t)}(z),$$

where $z = (z_1, \ldots, z_n)^T = (z_1, \xi)^T \in \mathbb{R}^n$, and $\phi_{\sigma(t)} : \mathbb{R}_+ \times \mathbb{Z} \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $z_1$, uniformly in $t$, on $\mathbb{R}_+ \times \mathbb{Z} \times \mathbb{R}^{n-1}$. Suppose that there exist functions $V_{2,n} : \mathbb{R}^{n-1} \to \mathbb{R}_+$ and $V_1 : \mathbb{Z} \to \mathbb{R}_+$, continuously differentiable and positive definite in their respective domains, such that

$$V_1(z_1) \to \infty \text{ as } z_1 \to -a \text{ or } z_1 \to b,$$

$$\gamma_1(\|\xi\|) \leq V_{2,n}(\xi) \leq \gamma_2(\|\xi\|),$$

where $\gamma_1$ and $\gamma_2$ are class $\mathcal{K}_\infty$ functions. Let $V(z) := V_1(z_1) + V_{2,n}(\xi)$, and $z_1(0)$ belong to the set $(-a, b)$. If the inequality holds:

$$\dot{V} \leq \frac{\partial V}{\partial z} \phi_{\sigma(t)} \leq 0,$$

then $z_1(t)$ remains in the open set $(-a, b), \forall t \in [0, \infty)$.

Lemma 2[21]. For any positive real numbers $c, d$ and any real-valued function $\rho(x, y) > 0$, 

$$|x|^c|y|^d \leq \frac{c}{c+d} \rho(x, y)|x|^{c+d} + \frac{d}{c+d} \rho^{-c/d}(x, y)|y|^{c+d}.$$ 

Lemma 3[28]. Barbalet’s Lemma) Consider a differentiable function $\eta(t)$. If $\lim_{t \to \infty} \eta(t)$ is finite and $\dot{\eta}$ is uniformly continuous, then $\lim_{t \to \infty} \dot{\eta} = 0$.

3 Main Result

3.1 A Constructive Design Method

In this section, we will give a constructive design method to obtain common stabilizing functions at each step and a CLF by the power integrators backstepping technique. Step 1. Let $z_1 := x_1 - y_r$, $z_2 := x_2 - \alpha_1(z_1)$, where $y_{lim} - y_r$, and $\alpha_1(z_1)$ is a stabilizing function to be designed. Choose the following symmetric Barrier Lyapunov Function candidate, originally proposed in [19]:

$$V_1(z_1) = \frac{1}{2} \ln\left(\frac{b^2}{b^2 - z_1^2}\right). \tag{3}$$

The derivative of $V_1(z_1)$ is given by

$$\dot{V}_1 = \frac{z_1}{b^2 - z_1^2} g_{1,k}(x_1)(z_2 + \alpha_1(z_1)).$$
Design the stabilizing function
\[ \alpha_1(z_1) = z_1 - \frac{1}{2} \left( \frac{1}{(b^2 - z_1^2)(n-2)+1} \right) + n(b^2 - z_1^2), \quad (4) \]

Substituting (4) into (3) yields:
\[
\dot{V}_1 = -\frac{z_1^2}{b^2 - z_1^2} \frac{g_1,k(x_1)}{g} n(b^2 - z_1^2) + \frac{1}{b^2 - z_1^2} g_1,k(x_1) z_1 z_2 \\
- \frac{z_1^2}{b^2 - z_1^2} \frac{g_1,k(x_1)}{g} (b^2 - z_1^2)(n-2) + \frac{1}{b^2 - z_1^2} g_1,k(x_1) z_1 z_2 \\
\leq -n z_1^2 - \frac{1}{b^2 - z_1^2} n(b^2 - z_1^2) + \frac{1}{b^2 - z_1^2} g_1,k(x_1) z_1 z_2 \\
\leq -n z_1^2 - \frac{1}{b^2 - z_1^2} (n-2) \frac{g_1,k(x_1)}{g} z_1 z_2 \\
\leq -n z_1^2 - \frac{1}{b^2 - z_1^2} (n-2)(b^2 - z_1^2) + \frac{1}{b^2 - z_1^2} g_1,k(x_1) z_1 z_2.
\]

where the coupling term \( \frac{1}{b^2 - z_1^2} g_1,k(x_1) z_1 z_2 \) will be canceled in the subsequent step.

**Step 2.** Let \( z_3 := x_3 - \alpha_2(z_2) \), where \( \alpha_2(z_2) \) is the stabilizing function to be designed.

Choose
\[ V_2(z_2) = V_1(z_1) + \frac{1}{2} z_2^2. \]

The time derivative of \( V_2(z_2) \) is given by:
\[
\dot{V}_2 \leq -n z_1^2 - \frac{1}{b^2 - z_1^2} (n-2) \frac{g_1,k(x_1)}{g} z_1 z_2 \\
+ z_2 \left( f_2,k(\tilde{x}_2) - \frac{\partial \alpha_1}{\partial z_1} \tilde{z}_1 + g_2,k(\tilde{x}_2) \right) \\
\leq -n z_1^2 - \frac{1}{b^2 - z_1^2} (n-2) \frac{g_1,k(x_1)}{g} z_1 z_2 \\
+ z_2 \Phi_2,k(z_2) + g_2,k(\tilde{x}_2) \left( z_3 + \alpha_2(z_2) \right)
\]

where \( \Phi_2,k(z_2) := f_2,k(\tilde{x}_2) - \frac{\partial \alpha_1}{\partial z_1} \Gamma_{1,k}(z_2), \ \Gamma_{1,k}(z_2) := g_1,k(x_1)(z_2 + \alpha_1(z_1)), \forall k \in M. \)

Using Assumption 1, we have
\[
|f_2,k(\tilde{x}_2)| \leq |x_2| \tilde{\mu}_{2,k}(\tilde{x}_2) \leq (|z_1| + |z_2|) \tilde{\mu}_{2,k}(\tilde{x}_2), \quad \forall k \in M,
\]

where \( \tilde{\mu}_{2,k}(\tilde{x}_2) \) are a set of smooth non-negative functions.

Then, it implies that
\[
|\Phi_2,k(z_2)| \leq (|z_1| + |z_2|) \tilde{\mu}_{2,k}(z_2), \quad \forall k \in M. \quad (5)
\]

Moreover, according to Lemma 2 and (5), it holds that
\[
|z_1 z_2| \leq z_1^2 + z_2^2 \varphi_2(z_2), \\
|z_2 \Phi_2,k(z_2)| \leq z_1^2 + z_2^2 \varphi_2(z_2), \quad \forall k \in M,
\]

where \( \varphi_2(z_2) \geq 1, \Phi_2,k(z_2) \geq 1 \) are some smooth functions. Therefore,
\[
\dot{V}_2 \leq -n z_1^2 - (n-2) \frac{g_1,k(x_1)}{g} z_1 z_2 \\
+ \frac{g_1,k(x_1)}{g} z_1 z_2 \varphi_2(z_2) + z_1^2 + z_2^2 \varphi_2(z_2) \\
+ g_2,k(\tilde{x}_2) z_2 \alpha_2(z_2) + g_2,k(\tilde{x}_2) z_2 z_3 \\
\leq -(n-1) z_1^2 - (n-2) \frac{g_1,k(x_1)}{g} z_1 z_2 \\
+ g_2,k(\tilde{x}_2) z_2 \alpha_2(z_2) + g_2,k(\tilde{x}_2) z_2 z_3
\]

where \( \varphi_{2,max}(z_2) \geq \varphi_2(z_2) \) satisfies:
\[
\alpha_2(z_2) := z_2 \left[ -\frac{1}{2} \left( \frac{\varphi_{2,max}(z_2)}{\varphi_2(z_2)} \right) + (n-2) \frac{g_1,k(x_1)}{g} + (n-1) \right]. \quad (7)
\]

Substituting (7) into (6) yields:
\[
\dot{V}_2 \leq -(n-1) z_1^2 - (n-2) \frac{g_1,k(x_1)}{g} z_1 z_2 + \frac{g_2,k(\tilde{x}_2) z_2 \alpha_2(z_2) + g_2,k(\tilde{x}_2) z_2 z_3}{\varphi_{2,max}(z_2)} \\
\leq -(n-1)(z_1^2 + z_2^2) - (n-2) \frac{g_1,k(x_1)}{g} z_1 z_2 + g_2,k(\tilde{x}_2) z_2 z_3,
\]

where \( \varphi_{2,max}(z_2) \geq 1, \Phi_{j,k}(z_2) := f_{j,k}(\tilde{x}_2) - \sum_{l=1}^{j-1} \frac{\partial \alpha_{l,j}}{\partial z_1} \Gamma_{l,k}(z_{l-1}), \)
\[
\Phi_{1,k}(z_1) := 0, \text{ we have a set of common stabilizing functions (4), (7) and}
\]
\[
\alpha_j(x_j) = z_j \left[ -\frac{1}{2} \left( \frac{\varphi_{j,2,max}(z_j)}{\varphi_2(z_2)} \right) + (n-j) \frac{g_1,k(x_1)}{g} + (n-j+1) \right]. \quad (9)
\]

Thus, there exists a common Lyapunov function for the transform system (8)
\[
V_{i-1}(z_{i-1}) = V_i(z_1) + \frac{1}{2} \sum_{l=2}^{i-1} z_l^2,
\]

and the time derivative of \( V_{i-1} \) satisfies:
\[
\dot{V}_{i-1} \leq -(n-i+2)(z_1^2 + \ldots + z_{i-1}^2) - (n-i+1) \frac{g_1,k(x_1)}{g} z_1 z_2 + g_{i-1,k}(\tilde{x}_{i-1}) z_{i-1} z_i.
\]

Choose
\[ V_i(z_i) = V_{i-1}(z_{i-1}) + \frac{1}{2} z_i^2, \]

then, we can obtain
\[
\dot{V}_i \leq -(n-i+2)(z_1^2 + \ldots + z_{i-1}^2) - (n-i+1) \frac{g_1,k(x_1)}{g} z_1 z_2 + g_{i-1,k}(\tilde{x}_{i-1}) z_{i-1} z_i \\
+ g_{i,k}(\tilde{x}_i) z_i \alpha_i(z_i) + g_{i,k}(\tilde{x}_i) z_i z_{i+1}.
\]
where \( \Phi_{i,k}(\bar{z}_i) := f_{i,k}(\bar{z}_i) - \sum_{n=1}^{N} \frac{\partial a_{i,n}}{\partial z_i} \Gamma_{i,k}(\bar{z}_{i-1}), \forall k \in M. \)

Similar to step 2, we have

\[
\begin{align*}
|z_{i-1}z_i| & \leq z_1^2 + \ldots + z_{i-1}^2 + z_i^2 \varphi_i(z_i), \\
|z_i \Phi_{i,k}(\bar{z}_i)| & \leq z_1^2 + \ldots + + z_i^2 \varphi_{i,k}(\bar{z}_i), \quad \forall k \in M.
\end{align*}
\]

where \( \varphi_i(z_i) \geq 1, \varphi_{i,k}(\bar{z}_i) \geq 1 \) are some smooth functions.

Therefore,

\[
\begin{align*}
\dot{V}_i & \leq -(n-i+2)(z_1^2 + \ldots + z_{i-1}^2) - (n-i+1) \bar{g}(z_1^2 + \ldots + z_{i-1}^2) \\
& \quad + \bar{g}z_1^2 + \ldots + \bar{g}z_{i-1}^2 + \bar{g}z_i^2 \varphi_i(z_i) + z_1^2 + \ldots + z_{i-1}^2 \\
& \quad + z_i^2 \varphi_{i,k}(\bar{z}_i) + g_{i,k}(\bar{x}_i) z_i \alpha_i(z_i) + g_{i,k}(\bar{x}_i) z_i z_{i+1} \\
& \leq -(n-i+1)(z_1^2 + \ldots + z_{i-1}^2) - (n-i) \bar{g}(z_1^2 + \ldots + z_{i-1}^2) \\
& \quad + z_i^2 \varphi_{i,max}(z_i) + g_{i,k}(\bar{x}_i) z_i \alpha_i(z_i) + g_{i,k}(\bar{x}_i) z_i z_{i+1} + 1,
\end{align*}
\]

where \( \varphi_{i,max}(z_i) \geq \varphi_{i,k}(z_i) := \bar{g} \varphi_i(z_1) + \varphi_{i,k}(\bar{z}_i) \geq \bar{g} + 1, \forall k \in M, \) is a smooth function.

Design the stabilizing function:

\[
\alpha_i(z_i) = z_1 \left[ -\frac{1}{2} (\varphi_{i,max}(\bar{z}_i) + (n-i) \bar{g} + (n-i+1)) \right], \quad (11)
\]

Then, substituting (11) into (10) yields:

\[
\begin{align*}
\dot{V}_i & \leq -(n-i+1)(z_1^2 + \ldots + z_{i-1}^2) - (n-i) \bar{g}(z_1^2 + \ldots + z_{i-1}^2) \\
& \quad + z_i^2 \varphi_{i,max}(z_i) - g_{i,k}(\bar{x}_i) \frac{\varphi_{i,max}(z_i)}{\bar{g}} - g_{i,k}(\bar{x}_i) (n-i) z_i z_{i+1} \\
& \quad - \frac{g_{i,k}(\bar{x}_i)}{\bar{g}} (n-i+1) z_i^2 + g_{i,k}(\bar{x}_i) z_i z_{i+1} \\
& \quad - (n-i) \bar{g}(z_1^2 + \ldots + z_{i-1}^2) \\
& \quad - (n-i) \bar{g}(z_1^2 + \ldots + z_{i-1}^2) + g_{i,k}(\bar{x}_i) z_i z_{i+1}. \\
\end{align*}
\]

where the coupling term \( g_{i,k}(\bar{x}_i) z_i z_{i+1} \) will be canceled in the subsequent step.

**Step n.** By using repeatedly the inductive argument above, it is straightforward to see that at the last step, there exists a common Lyapunov function

\[
V_n(\bar{z}_n) = V_1(z_1) + \frac{1}{2} \sum_{j=2}^{N} z_j^2, \quad (12)
\]

We can explicitly construct state feedback controllers for each subsystem:

\[
\begin{align*}
\dot{z}_n & = z_n \left[ -\frac{1}{g_{n,k}(\bar{x}_n)} (\varphi_{n,k}(\bar{z}_n) + 1) \right], \quad \forall k \in M, \\
\end{align*}
\]

such that

\[
\dot{V}_n \leq -(z_1^2 + \ldots + z_n^2). \quad (13)
\]

**Remark 2.** Obviously, we can also construct a smooth controller to stabilization the system (1) as:

\[
\begin{align*}
\dot{z}_n & = z_n \left[ -\dot{\bar{z}}_n \left( \varphi_{n,max}(\bar{z}_n) + 1 \right) \right], \quad \forall k \in M, \\
\end{align*}
\]

where \( \varphi_{n,max}(\bar{z}_n) \geq \varphi_{n,k}(\bar{z}_n) := \bar{g} \bar{\varphi}_n(\bar{z}_n) + \varphi_{n,k}(\bar{z}_n) \geq \bar{g} + 1 \) is a smooth function.

### 3.2 Output Tracking

In this subsection, we will give our main result.

**Theorem 1.** Consider the closed-loop system (1), (13) and suppose that Assumption 1-2 are satisfied. If the initial conditions are such that \( \bar{z}_n(0) \in \Psi_0 := \{ \bar{z}_n \in \mathbb{R}^n : |z_1| < b \} \), then the following properties hold.

(1) The signals \( z_i(t), i = 1, 2, \ldots, n \), remain in the compact set defined by

\[
\Psi_z = \{ \bar{z}_n \in \mathbb{R}^n : |z_1| \leq b \sqrt{1 - e^{-2V_0(0)}}, \forall t \geq 0 \}
\]

(2) All closed-loop signals are bounded.

(3) The output \( y(t) \) of system (1) asymptotically tracks the given constant signal \( y_r \), while, the constraint (2) is never violated.

**Proof.** (1) Since \( V_n \leq 0 \), hence \( V_n(t) \leq V_n(0) \). According to Lemma 1, we know that \( |z_1(t)| < b, \forall t > 0 \), provided that \( |z_1(0)| < b \).

Hence,

\[
\frac{1}{2} \ln \left( \frac{b^2}{b^2 - z_1^2(t)} \right) \leq 0.
\]

Then,

\[
\frac{b^2}{b^2 - z_1^2(t)} \leq e^{2V_0(0)}.
\]

Obviously, it holds that

\[
|z_1(t)| \leq b \sqrt{1 - e^{-2V_0(0)}}, \forall t \geq 0.
\]

Similarly, since

\[
\frac{1}{2} z_1^2(t) + \ldots + \frac{1}{2} z_n^2(t) \leq V_n(0),
\]

we can obtain the boundedness of \( z_2(t), \ldots, z_n(t) \).

Further, it is easy to have the estimate

\[
\|z_{2,n}(t)\| \leq \sqrt{2V_0(0) + n - 1}, \forall t \geq 0.
\]

Therefore, \( z_i(t) \) remains in the compact set \( \Psi_z, t \geq 0 \).

(2) From (1), we know that \( \bar{z}_n(t) \) are bounded. Obviously, from (4), the common stabilizing function \( \alpha_1(t) \) which is designed in the initial step is bounded. According to the relation \( x_2(t) = z_2(t) + \alpha_1(t) \), we can get the boundedness of \( x_2(t) \). From (14), \( |z_2(t)| \leq b \sqrt{1 - e^{-2V_0(0)}} < b \).

In addition, \( \alpha_2 \) is a continuous function of the bounded signals \( \bar{z}_2 \) in the set \( \bar{z}_2 \in (-b, b) \), it follows, from (7) that, the common stabilizing function \( \alpha_2(t) \) which is designed in the second step is bounded. Then, according to the relation \( x_3(t) = z_3(t) + \alpha_2(t) \), the boundedness of \( x_3(t) \) can be obtained.

Following this line of argument, we can progressively show that each stabilizing function \( \alpha_i(t), i = 3, \ldots, n - 1 \) is bounded, since it is a continuous function of the bounded signals \( \bar{z}_i(t) \) in the set \( \bar{z}_i \in (-b, b) \). Thus, according to the relation \( x_{i+1}(t) = z_{i+1}(t) + \alpha_i(t), i = 3, \ldots, n - 1 \), the boundedness of state \( x_{i+1}(t) \) can be obtained. Finally,
the boundedness of \( z_n(t) \) implies that the state feedback controllers \( u_{k}, k \in M \) for each subsystem is bounded. Therefore, all closed-loop signals are bounded. 

(3) In accordance to Lemma 3, it is easy to see that \( z(t) \to \infty \) as \( t \to \infty \), it implies that \( z_1(t) \to \infty \) as \( t \to \infty \), then \( y(t) \to y_r \) as \( t \to \infty \). In addition, \( |z_1(t)| < b, \forall t \geq 0 \), then it holds that \( |y(t)| = |z_1(t) + y_r| \leq |z_1(t)| + y_r < b + y_r = y_{lim} \).

**Remark 3.** From the above illustration, we know that the state \( x_1(t) \) is keeping in the boundary \( \pm \sqrt{1 - e^{-3t}x(0)} \) which is smaller than the allowable maximum boundary \( \pm b \). Obviously, there exist some safety margins in our design which is attributed to the Barrier Lyapunov Function.

**Remark 4.** From the proof of Theorem 1, it is clear that the state feedback controllers for each subsystem are only associated with the information of the bounding function \( \mu_{i,k}() \), \( i = 1, 2, \ldots, n \), \( \forall k \in M \) but the functions \( f_{i,k}(\cdot) \), that is, \( f_{i,k}(\cdot) \) need not to be known precisely, which implies that the controllers (13) possess some robustness. This characteristic is shown in the following subsection.

### 4 Example

We demonstrate the proposed backstepping designs using the following second order switched nonlinear system in p-normal form, satisfying Assumption 1:

\[
\Sigma_1 : \begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_2 \sin x_1 + u(x_1, x_2)
\end{cases}, \quad (15)
\]

\[
\Sigma_1 : \begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = \frac{1}{2}x_1 + u(x_1, x_2)
\end{cases}, \quad (16)
\]

The reference signal \( y_r = 2 \), our control objective is to design a smooth controller to ensure that the output \( y(t) \) of the switched system track the given signal, while it doesn’t destroy the constraint \( |y(t)| < 3, \forall t \geq 0 \).

Using the design method presented in Section 3, let \( z_1 = x_1 - 2 \), one can obtain the common stabilizing function \( \alpha(z_1) \) for each subsystem at the initial step:

\[
\alpha(z_1) = -z_1(3 - 2z_1^2)
\]

Let \( z_2 = x_2 - \alpha(z_1) \), we can design the state-feedback controllers as follows:

\[
u(z_1, z_2) = \frac{1}{2}[(3 - 2z_1^2)(13 + 3z_2^2 - 6z_1^2)]^2 + (3 - 2z_1^2)(13 + 3z_2^2 - 6z_1^2) + \frac{1}{1 - z_1^2}.
\]

and the CLF is given by

\[
V(z_1, z_2) = \frac{1}{2} \ln\left(\frac{1}{1 - z_1}\right) + \frac{1}{2} z_2^2.
\]

The simulation result is depicted in Fig.2 which is shows the asymptotic stability of the closed-loop system (15),(16),(17) with the initial state \( z_1(0) = -0.8, z_2(0) = 1 \) under the switching signal depicted in Fig.3. Furthermore, from Fig.2, it is easy to see that the output-constrained \( y(t) \) is keeping in the boundary \( \pm 3 \) all the time.

![Fig. 2: The constrained output y(t).](image)

![Fig. 3: Switching signal.](image)

### 5 Conclusion

In this paper, the tracking problem for a class of output-constrained switched nonlinear system in strict-feedback form is investigated. There are three points that deserves to be summarized again. Firstly, we introduce the Barrier Lyapunov Function to deal with the constrained problem for the output constraint, and employ the traditionary Quadratic Lyapunov Function for the free-constrained states. Secondly, we explicitly construct the common stabilizing functions at each step, and then, we obtain a CLF for each subsystem which implies that the closed-loop switched nonlinear system is stabilizable under arbitrary switchings. Thirdly, the state feedback controllers for each subsystem possess robustness for some unknown disturbance belonging a known compact set. Finally, an example illustrates the effectiveness of the proposed method.

### References


[27] L.J.Long and J.Zhao, “Global stabilization of switched nonlinear systems in p-normal form with mixed odd and even pow-