Exponential Stability for Switched Delay Systems Based on Average Dwell Time Technique and Lyapunov Function Method

Xi-Ming Sun, Georgi M. Dimirovski, Jun Zhao, and Wei Wang

Abstract—This paper considers the problem of guaranteed exponential stability of switched delay systems by using Lyapunov function method, Razumikhin technique. The scheme of average dwell time is introduced into the switched delay systems. Based on the scheme of average dwell time, sufficient conditions for exponential stability of switched delay systems are presented. For a certain class of switched delay systems, a lower bound on the average dwell time guaranteeing exponential stability can be explicitly calculated via the solutions of some linear matrix inequalities (LMIs). A numerical example is given to illustrate the effectiveness of the proposed method.

I. INTRODUCTION

Time delay is a common phenomenon encountered in engineering systems. Because delays are frequently a source of instability and often deteriorate system performances, it is of great importance to study the stability issue for delay systems. Lyapunov methods including Lyapunov functional method [1-5, 19] and Lyapunov-Razumikhin method [6-7, 19] are still dominating methods.

On the other hand, switched systems are a special class of hybrid control systems, which consist of a family of continuous-time or discrete-time subsystems and a switching law that orchestrates the switching between them. Such systems have drawn considerable attention in control and computer communities in the last decade [8-11]. Dwelling based approaches provide an effective tool to study the stability of switched systems. It was shown in [12] that a switched system is exponentially stable if all subsystems are stable and the dwell time is set sufficiently large. Subsequently, in [13] the concept of dwell time is extended to average dwell time. Furthermore, in [14], the stability results are extended to the case where stable and unstable subsystems co-exist.

Switched systems with time delay are referred to as switched delay systems, which are a brand new type of systems and often appear in the modellings of Nakagami-fading systems [15], networked control systems [16] and so on. At present, the results on switched delay systems have a few[17-18]. Since for switched systems without delays, the average dwell time technique is an important method. This technique is believed to be more desirable and preferable to analyze stability of switched delay systems. However, to our best knowledge, there have been no such results available up to now. In addition, it is known that in some practical applications, the asymptotical stability can not meet engineering requirements, and exponential stability with certain decay degree is preferable. Unfortunately, also no results on exponential stability analysis for switched delay systems have been reported till now.

This paper deals with the problem of exponential stability analysis for switched delay systems. Sufficient conditions for exponential stability of switched delay systems through the average dwell time technique and Lyapunov function method. For a certain class of switched delay systems, a lower bound on the average dwell time is explicitly calculated via the solutions of some linear matrix inequalities (LMIs) problem. Moreover, the state decay estimate is also easily obtained.

This paper is organized as follows. In Section 2, some preliminary knowledge and problem formulation are introduced. Two main results about exponential stability for switched delay systems are presented in Section 3. An example is given in Section 4. Section 5 gives a conclusion.

II. PRELIMINARY KNOWLEDGE AND PROBLEM FORMULATION

We first introduce some notations. Let \( R_+ = [0, +\infty) \), \( I_n \) denotes \( n \)-dimensional identity matrix. Let \( \lambda_{\min}(\cdot)(\lambda_{\max}(\cdot)) \) denote the minimum (maximum) eigenvalue of matrix \( \cdot \). For given \( \tau \geq 0 \), \( C_n = C([-\tau, 0], \mathbb{R}^n) \) denotes the Banach Space of continuous functions mapping from \([-\tau, 0], \mathbb{R}^n\) to \( \mathbb{R}^n \) with topology of uniform convergence. Let \( x_t \in C_n \) be defined by

\[
x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0].
\]

\( ||x(t)|| \) denotes usually 2-norm and

\[
||x_t|| = \sup_{-\tau \leq \theta \leq 0} ||x(t + \theta)||.
\]

Let \( V : R_+ \times \mathbb{R}^n \rightarrow R_+ \) be a continuous function. \( \dot{V}(t,x(t)) \) is defined as

\[
\dot{V}(t,x(t)) = \sup_{\theta \in [-\tau,0]} \{ V(t+\theta,x(t+\theta)) \},
\]

and let

\[
D^+V(t,x(t)) = \lim_{h \to 0^+} \frac{1}{h} [V(t+h,x(t+h)) - V(t,x(t))], t \geq t_0.
\]
For convenience, we denote

\[ V(t) = V(t, x(t)), \tilde{V}(t) = \tilde{V}(t, x(t)) . \]

Consider a class of switched delay systems of the form

\[
\begin{align*}
\dot{x}(t) &= f_{\sigma(t)}(t, \phi(t)), \\
x_0(\theta) &= \phi(\theta), \theta \in [-\tau, 0],
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state, \( \sigma : [0, +\infty) \to M = \{1, 2, \ldots, m\} \) is a piecewise constant function which depends on time \( t \) or state \( x(t) \); \( \phi(\theta) \in C_\alpha \) is vector-value initial condition. For any \( p \in M, f_p : \mathbb{R}_+ \times C_\alpha \to \mathbb{R}_+ \) is continuous and sufficiently smooth. We assume that \( f_p(t, 0) \equiv 0 \) for all \( t \) so that \( x(0) = 0 \) is an equilibrium of system (1). Corresponding to the switching signal \( \sigma \), we have the switching sequence

\[ \sum = \{ x_0; (i_0, t_0), (i_1, t_1), \ldots, (i_k, t_k), \ldots, |i_k \in M, k \in N \}, \]

when \( t \in [t_k, t_{k+1}] \), the \( i_k \)-th subsystem is activated. We assume that the state of the switched system (1) does not jump at the switching instants, i.e., the trajectory \( x(t) \) is everywhere continuous.

**Definition 1:** The equilibrium \( x^* = 0 \) of system (1) is said to be globally exponentially stable if the solution \( x(t, \phi)(t) \) of system (1) through any \( (t_0, \phi) \in \mathbb{R}_+ \times C_\alpha \) satisfies

\[ \| x(t_0, \phi)(t) \| \leq \Gamma \| x_0 \| e^{-\gamma(t-t_0)}, \forall t \geq t_0, \]

where \( \Gamma > 1 \) and \( \gamma > 0 \) are constant. In this case, we often say system (1) is globally exponentially stable.

**Definition 2:** For any switching signal \( \sigma \) and any \( T_2 > T_1 \geq 0 \), let \( N_\sigma(T_1, T_2) \) denote the number of switchings of \( \sigma \) over the internal \([T_1, T_2] \), and for given \( T_0 > 0 \), if

\[ N_\sigma(T_1, T_2) \leq \frac{T_2 - T_1}{T_0}, \]

the positive constant \( T_0 \) is referred to as average dwell time.

**Problem formulation:** Under the assumption that all the subsystems of system (1) are exponentially stable, we find a lower bound on average dwell time such that the system (1) is globally exponentially stable.

When \( M = 1 \), switched delay system (1) reduces to the usual delay systems:

\[
\begin{align*}
\dot{x}(t) &= f(t, x_t), \\
x_0(\theta) &= \phi(\theta), \theta \in [-\tau, 0].
\end{align*}
\]

For this system, we have the following Lemma.

**Lemma 1:** Let \( V : [-\tau, +\infty) \times \mathbb{R}^n \to \mathbb{R}_+ \) be a continuous function. Along the solution of system (2) through any \( (t_0, \phi) \in \mathbb{R}_+ \times C_\alpha \), we have

\[ \tilde{V}(t) \leq \tilde{V}(t_0) e^{-\gamma(t-t_0)}, \forall t \geq t_0, \]

if and only if

\[ D^+ \tilde{V}(t) \leq -\gamma \tilde{V}(t), \]

whenever

\[ \begin{align*}
\tilde{V}(t) &\leq \tilde{V}(t_0) e^{\gamma t}, \\
\tilde{V}(t) &= \tilde{V}(t_0) e^{-\gamma(t-t_0)}. \end{align*} \]

**Proof:** It is an immediate result of Theorem 1 in [6].

**III. MAIN RESULTS**

**Theorem 1:** For any \( p \in M \), let

\[ V_p : [-\tau, +\infty) \times \mathbb{R}^n \to \mathbb{R}_+ \]

be continuous functions. Suppose there exist positive constants \( a, b, \gamma \) and \( \mu \geq 1 \) such that the following hold

\[ a \| x(t) \|^2 \leq V_p(x(t)), \tilde{V}_p(x(t_0)) \leq b \| x_0 \|^2, \forall x \in \mathbb{R}^n, \forall p \in M, \forall t \geq t_0, \]

\[ D^+ V_p(t) \leq -\gamma V_p(t), \]

whenever

\[ \begin{align*}
\tilde{V}_p(t) &\leq \tilde{V}_p(t_0) e^{\gamma t}, \\
\tilde{V}_p(t) &= \tilde{V}_p(t_0) e^{-\gamma(t-t_0)}. \end{align*} \]

and

\[ V_p(x) \leq \mu V_p(x), \forall x \in \mathbb{R}^n, \forall p, q \in M. \]

Then the switched delay system (1) is globally exponentially stable for every switching signal \( \sigma \) with average dwell time \( T_0 > \frac{\ln(\mu + 1)}{\gamma} \), and the state decay estimate is given by

\[ \| x(t) \| \leq \sqrt{\frac{b e^{\gamma T_0}}{a}} \| x_0 \| e^{-\gamma \frac{\ln(\mu + 1)}{\gamma}(t-t_0)}. \]

**Proof:** For \( t \in [t_k, t_{k+1}] \), considering (4), (5) by Lemma 1, we can easily get

\[ \tilde{V}(t_k) \leq \tilde{V}(t_{k-1}) e^{-\gamma(t-t_{k-1})}. \]

From (6), we have

\[ \tilde{V}(t_k) \leq \mu \tilde{V}(t_{k-1}) e^{-\gamma(t-t_{k-1})}, \]

which yields

\[ \tilde{V}(t) \leq \mu^k \tilde{V}(t_{k-1}) e^{-\gamma(t-t_{k-1})}. \]

Similarly, for interval \([t_{k-1}, t_k]\), we have

\[ \tilde{V}(t_{k-1}) \leq \tilde{V}(t_{k-2}) e^{-\gamma(t-t_{k-2})}. \]

Thus, we have

\[ \tilde{V}(t_k) \leq \mu^k \tilde{V}(t_{k-1}) e^{-\gamma(t-t_{k-1})} \]

\[ \leq \mu \tilde{V}(t_{k-1}) e^{-\gamma(t-t_{k-1})} e^{-\gamma(t-t_{k-2})} \]

\[ \leq \mu^k \tilde{V}(t_{k-1}) e^{-\gamma(t-t_{k-1})} \]

\[ = e^{-\gamma \frac{\ln(\mu + 1)}{\gamma}(t-t_{k-1})} \tilde{V}(t_{k-1}). \]

Thus we obtain

\[ \| x(t) \|^2 \leq \frac{1}{a} \tilde{V}(t) \leq \frac{1}{a} \tilde{V}(t_k) \]

\[ \leq \frac{b}{a} e^{-\gamma \frac{\ln(\mu + 1)}{\gamma}(t-t_{k-1})} e^{\gamma T_0} \tilde{V}(t_{k-1}). \]

We get

\[ \| x(t) \|^2 \leq \frac{b}{a} e^{-\gamma \frac{\ln(\mu + 1)}{\gamma}(t-t_{k-1})} e^{\gamma T_0} \| x_0 \|^2, \]
which directly leads to (7). Thus, when \( \gamma > \frac{\ln \mu + \gamma \tau}{\tau_0} \), that is, average dwell time satisfies
\[
T_0 > T_u^{\ast} = \frac{\ln \mu + \gamma \tau}{\gamma},
\]
the system (1) is globally exponentially stable. The proof is completed.

**Remark 1:** If \( \tau = 0 \), the switched delay system (1) reduces to the usual switched systems without delay. The average dwell time just corresponds to one in [8]. So, Theorem 1 extends the average dwell time approaches for usual switched systems without delays in [8].

Now, let us consider the following switched delay systems with special structure
\[
\dot{x}(t) = A_{\sigma(t)}(t)x(t) + F_{\sigma(t)}(t,x_t), \quad x_0 = \varphi(\theta), \theta \in [-\tau,0],
\]
where, for \( \forall p \in M, A_p(t) \in \mathbb{R}^{n \times n} \) are continuous and bounded on \( \mathbb{R}_+ \), \( F_p : \mathbb{R}_+ \times \mathbb{C}^n \to \mathbb{R}^n \) are continuous and satisfy
\[
\|F_p(t,x_t)\| \leq H_p \|x_t\|,
\]
\( H_p \) are positive numbers.

When \( M = 1 \), the system (8) becomes the following delay system
\[
\dot{x}(t) = A(t)x(t) + F(t,x_t), \quad x_0 = \varphi(\theta), \theta \in [-\tau,0],
\]
where
\[
\|F(t,x_t)\| \leq H \|x_t\|,
\]
\( H \) is a positive constant.

Before we arrive at the other main result, we introduce the following Lemma.

**Lemma 2:** For system (9), let
\[
V(t) = x^T(t)P(t)x(t),
\]
suppose there exist positive constants \( \lambda_0, \alpha, \beta \), symmetric matrices \( P(t) \) and \( Q(t) \) such that
\[
\lambda_0 I_n \leq Q(t), \quad \alpha I_n \leq P(t) \leq \beta I_n,
\]
and
\[
\dot{P}(t) + P(t)A(t) + A^T(t)P(t) = -Q(t).
\]
Also suppose that the equation
\[
\lambda_0 - 2\beta \left( \frac{\beta}{\alpha} \right)^{\frac{1}{2}} \|x\| \exp\left( \frac{1}{2} \lambda_0 \tau \right) = \gamma_0 \beta
\]
has a unique constant solution \( \gamma_0 > 0 \).

Then we have
\[
\hat{V}(t) \leq \hat{V}(t_0)e^{-\gamma \tau(T-T)}, \forall t \geq t_0, t_0 \geq 0.
\]

**Proof:** By making use of Lemma 1, the proof is similar to that of Theorem 3 in [6] and thus is omitted.

For system (8), we have the following result.

**Theorem 2:** Suppose for \( \forall p \in M \), there exist positive constants \( \lambda_p, \alpha_p \) and \( b_p \), symmetric matrices \( P_p(t) \) and \( Q_p(t) \) such that
\[
\lambda_p I_n \leq Q_p(t), \quad \alpha_p I_n \leq P_p(t) \leq b_p I_n.
\]
Also suppose that the equation
\[
\lambda_p - 2b_p \left( \frac{b_p}{\alpha_p} \right)^{\frac{1}{2}} \|x\| \exp\left( \frac{1}{2} \lambda_p \tau \right) = \gamma_p
\]
has a unique positive constant solution \( \gamma_p > 0 \), and there exists a constant \( \mu \geq 1 \) such that
\[
P(t) \leq \mu P(t), T_0 \geq 0, \forall t \in M.
\]
Then the system (8) is globally exponentially stable for every switching signal \( \sigma \) with average dwell time \( T_0 > T_u^{\ast} \), and the state decay estimate is given by
\[
\|x(t)\| \leq \sqrt{\frac{\ln \mu + \gamma \tau}{a}} \exp\left( \frac{1}{2} \gamma_0 \tau \|x_0\| \right),
\]
where
\[
a = \min_{p \in M} \{ \alpha_p \}, b = \max_{p \in M} \{ b_p \}, \quad \gamma = \min \{ \gamma_1, \gamma_2, \ldots, \gamma_p \}.
\]

**Proof:** For any \( p \in M \), let
\[
V_p(t) = x^T(t)P_p(t)x(t),
\]
where \( P_p(t) \) is the solution of equation (13) and satisfies (12). For \( t \in [t_k, t_{k+1}) \), from the condition (12), (13) and (14), by using Lemma 2, we have
\[
\hat{V}_p(t) \leq \hat{V}_p(t_k)e^{-\gamma(t-t_k)}, \forall t \geq 0, \forall p \in M.
\]
Then, we have
\[
\hat{V}_p(t) \leq \hat{V}_p(t_k)e^{-\gamma(t-t_k)}.
\]
Thus, similarly to the proof of Theorem 1, we easily get
\[
\hat{V}_\sigma(t) \leq \mu^{\hat{V}}_{\sigma(t_k)}(t_k)e^{-\gamma(t-t_k)}
\]
\[
\ldots
\]
\[
\leq \mu^k \hat{V}_{\sigma(t_k)}(t_k)e^{k\gamma \tau}e^{-\gamma(t-t_k)}
\]
\[
\leq \mu^{\frac{t-t_0}{T_0}} e^{-\gamma(T-T)} \hat{V}_{\sigma(t_0)}(t_0)
\]
\[
= e^{-\gamma \frac{t-t_0}{T_0}} e^{\gamma \tau} \hat{V}_{\sigma(t_0)}(t_0).
\]
From (12), we know
\[ a\|x(t)\|^2 \leq \dot{V}_p(x(t)) \leq b\|x_t\|^2, \forall p \in M. \]
where \(a\) and \(b\) are defined above.
Thus we have
\[
\|x(t)\|^2 \leq \frac{1}{a} \dot{V}_p(\sigma(t)) \leq \frac{1}{a} e^{-(\frac{\ln a + \gamma \tau}{\mu})(t-t_0)} e^{T\tau} \bar{V} \sigma(t_0) \leq \frac{b e^{\gamma \tau}}{a} e^{-(\frac{\ln a + \gamma \tau}{\mu})(t-t_0)} \|x_0\|^2,
\]
which is just (16). So, when \(\gamma > \frac{\ln a + \gamma \tau}{\mu}\), that is, average dwell time satisfies
\[
T_a > T_a^\ast = \frac{\ln \mu + \gamma \tau}{\gamma},
\]
the system (8) is globally exponentially stable. The proof is completed.

If \(A_p(t), \forall p \in M\) are constant matrices, then the system (8) reduces to the following form
\[
\dot{x}(t) = A_{\sigma(t)} x(t) + F_{\sigma(t)}(t, x_t),
\]
where \(\sigma(t) = \sigma(\theta), \theta \in [-\tau, 0]\).
For system (18), according to Theorem 2, we have the following Corollary.

**Corollary:** For any \(p \in M\), suppose there exist positive definite matrices \(P_p\) and positive constants \(\lambda_p\) such that LMIs
\[
P_p A_p + A_p^T P_p + \lambda_p I < 0, \forall p \in M
\]
hold and for each \(P \in M\), the following equation
\[
\lambda_p = 2b_p \left( \frac{a_p}{\mu} \right)^2 H_p \exp \left\{ \frac{1}{2} \gamma_p \tau \right\} = \gamma_p b_p
\]
has a unique positive constant solution \(\gamma_p > 0\). Also suppose there exists a constant \(\mu \geq 1\) such that
\[
P_p \leq \mu P_q, \forall p, q \in M.
\]
Then the system (18) is globally exponentially stable for every switching signal \(\sigma\) with average dwell time
\[
T_a > T_a^\ast = \frac{\ln \mu + \gamma \tau}{\gamma},
\]
Moreover, the state decay estimate for the system (18) is given by (16), where \(\gamma = \min_{p \in M} \{\gamma_p\}\), \(a_p\) and \(b_p\) are all positive constants defined by
\[
a_p = \lambda_{\min}(P_p), b_p = \lambda_{\max}(P_p).
\]

**Proof:** According to [6], conditions (12) and (13) reduce to LMI (19) for system (18). Thus the result follows from Theorem 2 immediately. This completes the proof.

**Remark 2:** If all state matrices \(A_p, \forall p \in M\) are Hurwitz stable, then LMIs (19) hold and inequality (21) will be guaranteed for a certain \(\mu\). In this case, exponential stability of system (18) will be ensured once the solutions of equation (20) exist.

**Remark 3:** According to Corollary, the following steps provide the convenient methods in the sense of estimating exponential stability for system (18).

Step 1. Given \(\mu \geq 1\), Solving the LMIs (19) and (21) to obtain positive definite matrices \(P_p\) and positive constants \(\lambda_p\). Then let
\[
a_p = \lambda_{\min}(P_p), b_p = \lambda_{\max}(P_p).
\]

Step 2. Solve equation (20) to obtain \(\gamma_p\) and \(\gamma\).

Step 3. According to (22), we will obtain the lower bound on average dwell time \(T_a\), and state estimate is given by (16).

### IV. Numerical Example

Consider the following switched delay system
\[
\dot{x}(t) = A_{\sigma(t)} x(t) + E_{\sigma(t)} x(t-\tau),
\]
where \(\sigma(t) = \{1, 2\}, A_1 = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}, \)
\[
E_i = \begin{pmatrix} 0.8746 & 0.6721 \\ 0.6721 & 2.6905 \end{pmatrix}, P_i = \begin{pmatrix} 0.8698 & 0.6717 \\ 0.6717 & 2.6811 \end{pmatrix},
\]
\[
\lambda_1 = 2.0517, \lambda_2 = 2.4778.
\]
Then we can get
\[
a_1 = \lambda_{\min}(P_1) = 0.6530, b_1 = \lambda_{\max}(P_1) = 2.9121,
\]
\[
a_2 = \lambda_{\min}(P_2) = 0.6479, b_2 = \lambda_{\max}(P_2) = 2.9030.
\]
Solving equation (20) leads to
\[
\gamma_1 = 0.0985, \gamma_2 = 0.2263.
\]
Then
\[
\gamma = \min\{\gamma_1, \gamma_2\} = 0.0985.
\]
So, average dwell time
\[
T_a > T_a^\ast = \frac{\ln 1.02 + 0.0985 \times 0.5}{0.0985} = 0.7010.
\]
Then switched delay systems (23) is globally exponentially stable with average dwell time \(T_a\). Taking \(T_a = 0.9 > 0.7010\), we obtain state decay estimate of the system (23) according to (16)
\[
\|x(t)\| \leq 2.1729 e^{-0.0109(t-t_0)} \|x_0\|.
\]
V. Conclusion

This paper has successfully dealt with the problem of exponential stability for switched delay systems. Based on average dwell time scheme, we adopt Lyapunov function method to establish the sufficient condition of exponential stability for switched delay systems. For a class of switched delay systems, a lower bound on average dwell time can be easily obtained and the state decay estimate can also be calculated by making use of the solutions LMI's. It should be noted, however, directly using Razumikhin-type of Lyapunov function method may also cause some conservatism. Thus, in the future, it is necessary to make use of Lyapunov functional method or improved Razumikhin-type results for studying the exponential stability of switched delay systems.

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VII. References