QUADRATIC STABILITY OF A CLASS OF SWITCHED SYSTEMS

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Abstract

Quadratic stability of a class of switched non-linear systems is studied in this paper. We first transform quadratic stability problem into an equivalent nonlinear programming problem. Then, with the help of Fritz John condition for nonlinear programming problems, we derive a necessary and sufficient condition for switched systems to be quadratically stable. This necessary and sufficient condition is consisted of a set of algebraic equations and inequalities that are amenable to computer algebra evaluation.

1 Introduction

Complex non-linear systems involving dynamics governed by both logic symbol strings and real valued functions have been noted for quite some time, and the need for hybrid representation modelling emphasised [3], [5], [8], [10], [15], [17]. However, considerable progress in theoretical study of hybrid systems has been made during the recent decade only [4], [6], [7], [9], [11]-[14], [19]-[24]. Recently the switched systems, as a special kind of hybrid systems, have been in a focus of interest, in particular [4], [7], [15], [17], [21], [23].

Typically, a switched system is consisted a number of subsystems, which are ordinary continuous-time or discrete-time systems, and a switching law, which defines a specific subsystem being activated during certain time interval. Switched systems arise in many areas of engineering applications. To name a few, for instance, typical examples of switched systems can be found in constrained robotics [1], computer-controlled devices in motion control such as stepper motors [6], computer disk drives [11], automated highway systems [21], the well-known cart-pendulum system [24], etc.

Switched systems also arise in applications of multiple controllers for controlling complex non-linear processes. These have been widely used in adaptive control where a higher-level, logic-based supervisor provides switching between a family set of candidate controllers so as to achieve desired performance of the closed-loop system. As special types of switched systems, switched controller systems often give satisfactory solution guaranteeing stability and good performance when no single controller is effective [14], [17]. This is true even when the plant is poorly modelled or
many constrains on control, state and output exist [8], [10], [15].

The stability is known to be the most central issue in the study of switched systems as is the case of all non-linear and hybrid dynamical systems [2], [22]. There have been a number of works in this direction, e.g., see [5], [12], [13], [19], [20]. The existence of a common Lyapunov function for all subsystems is a necessary and sufficient condition for a switched system to be asymptotically stable under arbitrary switching [15]. Indeed, there exist many methods to construct a Lyapunov function (see, for example, [2], [5], [19]). However, most switched systems do not possess a common Lyapunov function and, nonetheless, they still may be asymptotically stable under a properly chosen switching law. Single and multiple Lyapunov function techniques are effective tools to find such a switching law [5], [12], [13].

Quadratic stability is known to be a preferable system property. Many switched systems that are asymptotically stable, however, are not necessarily quadratically stable too, unlike the case of ordinary linear systems for which quadratic stability is equivalent to asymptotic stability. The latter does not hold true even for switched linear systems. Hence special tools to deal with quadratic stability for switched systems are most needed. To the best of our knowledge, very few results on quadratic stability of switched non-linear systems have been reported. Quadratic stability in the case of piece-wise linear systems was studied in [13]. Quadratic stability of switched systems is equivalent to the completeness of certain set of functions generated by a positive definite matrix [20]. However, the completeness is extremely difficult to verify in general.

The purpose of this paper is to lay down the theoretical grounds for dealing with the quadratic stability problem in an alternative way, different than the established ones. Namely, we first transform quadratic stability problem into an equivalent nonlinear programming problem. Then, with the help of Fritz John condition for nonlinear programming problems, a necessary and sufficient condition for switched nonlinear systems to be quadratically stable is derived. This condition consists of a set of algebraic equations and inequalities, and thus considerably easy to apply and verify.

2 On Certain Preliminaries

In the study of stability of switched systems, the concept of completeness has been found rather useful.

Definition 2.1 [20]. A set of continuous functions \( \{v_1, v_2, \ldots, v_k\} \) where \( v_j: \mathbb{R}^n \to \mathbb{R} \), is called complete if for any \( x \in \mathbb{R}^n \), there exists an \( i \in \{1, 2, \ldots, k\} \) such that \( v_i(x) \leq 0 \). In addition, the set \( \{v_1, v_2, \ldots, v_k\} \) is called strictly complete if for any \( x \neq 0 \) there exists an \( i \in \{1, 2, \ldots, k\} \) such that \( v_i(x) < 0 \).

The concept of (strict) completeness of functions is a generalization of the concept of (negative) non-positive definite functions.

Let now consider the switched non-linear system

\[
\dot{x} = f_i(x), \quad i = 1, 2, \ldots, k
\]

where \( x \in \mathbb{R}^n \). As a straightforward consequence of the direct Lyapunov method, the asymptotic stability of system (1) can follow from the completeness of a properly chosen set of functions. This has given rise to the following theorem.

Theorem 2.1 [20]. System (1) is asymptotically stable if there exists a positive definite radially unbounded function \( V(x) \), such that the set of functions \( \{L_{f_j}V \} \) is strictly complete, where \( L_{f_j}V \) is the Lie derivative of \( V \) along \( f_j \), i.e., \( L_{f_j}V = \frac{\partial V}{\partial x} f_j \).

This result means that strict completeness of certain set of functions generated by a positive definite matrix \( P \) implies quadratic stability. However, unfortunately it is usually very difficult to check the strict completeness. Our goal in this work is to find an alternative condition. In order to do this, the Fritz John condition for nonlinear programming problems is needed.

Lemma 2.1 [3]. Given a nonlinear programming problem
3 Quadratic Stability of Switched Systems

In this study, we investigate the quadratic stability of switched non-linear systems only. Therefore, in the sequel we first give a known set of related precise definitions and lemmas, based on the literature, before proceeding to elaborate our approach to the quadratic stability.

Definition 3.1. Switched non-linear system (1) is said to be quadratically stable if there exist a positive definite matrix P and a switching law with the “state feedback” form i = i(x) such that the derivative of the function \( V = x^T P x \) is negative along the trajectory of system (1) for any \( x \neq 0 \).

For the quadratic stability of system (1) we have the following result, which can be easily proved by means of Theorem 2.1.

Lemma 3.1. System (1) is quadratically stable if and only if a positive definite matrix P exists such that the set of functions \( \{ x^T P f_j(x), j = 1, 2, \ldots, k \} \) is strictly complete.

It is still very difficult to verify quadratic stability by directly applying Lemma 3.1. We will limit ourselves to specific switched non-linear systems that are much related to the concept of generalized homogeneity of continuous mappings.

Definition 3.2. A continuous mapping \( f(x): \mathbb{R}^n \rightarrow \mathbb{R}^k \) is said to be generalized homogeneous if there exists a positive function \( \gamma(x, \alpha): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}_+ \) such that for any \( \alpha > 0, \ x \neq 0, \ f(\alpha x) = \gamma(x, \alpha) f(x) \) holds.

Remark 3.1. If \( f(x) \) is a homogeneous mapping with any degree \( \rho \), that is, \( f(\alpha x) = \alpha^\rho f(x) \) for all \( \alpha > 0, \ x \neq 0 \), then \( f(x) \) is certainly a generalized homogeneous mapping with \( \gamma(x, \alpha) = \alpha^\rho \). In particular, linear and bilinear mappings all are generalized homogeneous.

Lemma 3.2. A generalized homogeneous mapping \( f(x) \) must have the property that \( f(x) = 0 \) if and only if \( f(\alpha x) = 0 \) for all \( \alpha > 0 \).

Now, let focus the attention on this lemma and proceed further. Suppose P is a positive definite matrix. For any fixed \( i \in \{ 1, 2, \ldots, k \} \) and \( x \in \mathbb{R}^n, \ x \neq 0 \), define the set

\[
M_i(x) = \left\{ j: x^T P f_j(x) = 0, j \neq i \right\},
\]

and the vector fields

\[
H_j = P f_j + \frac{\partial f_j}{\partial x} x^T P x - x^T \frac{\partial f_j}{\partial x} P x, j = 1, 2, \ldots, k.
\]

If \( H_j, j = 1, 2, \ldots, k, \) are not all zero at \( x \), choose one of the largest linearly independent subsets of vectors in \( \{ H_j, j \in M_i(x) \} \) (empty set if \( H_j = 0, j \in M_i(x) \)), denoted by \( H_{j_1}, \ldots, H_{j_s} \). Let define

\[
H = (H_{j_1}, \ldots, H_{j_s}), S_i = \{ j_1, \ldots, j_s \}, Q_i = M_i \setminus S_i.
\]

In what follows below, in order to avoid checking directly the strict completeness, we transform it into a nonlinear programming problem. This is done by means of the following theorem.

Theorem 3.1. Suppose \( x^T P f_j(x) \) is generalized homogeneous, \( j = 1, \ldots, k \). System (1) is quadratically stable with respect to the quadratic Lyapunov function \( x^T P x \) if and only if an \( i \in \{ 1, 2, \ldots, k \} \) exists which makes the nonlinear programming problem

\[
\begin{align*}
\text{Max} & \quad f(x) \\
\text{s.t.} & \quad s_j(x) \geq 0, j = 1, 2, \ldots, m, \\
& \quad h_j(x) = 0, j = 1, 2, \ldots, l,
\end{align*}
\]

where \( f, s_j, h_j : \mathbb{R}^n \rightarrow \mathbb{R} \) are smooth functions. If \( x^* \) is a local optimal solution to the problem, then there exist constants \( \mu_0, \mu_1, \ldots, \mu_m \), not all zero, and constants \( l_1, l_2, \ldots, l_l \) such that

\[
\begin{align*}
\mu_0 & \nabla f(x^*) + \sum_{i=1}^m \mu_i s_i(x^*) + \sum_{i=1}^l l_i h_i(x^*) = 0, \\
\mu_i s_i(x^*) & = 0, i = 1, 2, \ldots, m, \\
\mu_i & \geq 0, i = 1, 2, \ldots, m,
\end{align*}
\]

where \( \nabla \) stands for the gradient operator.
Max \[ x^T Pf_j(x) \]
subject to \[ x^T Pf_j(x) \geq 0, \quad j \neq i, \]
\[ x^T x - 1 = 0 \]
(6)

have no feasible solution or the optimal value is negative if a feasible solution exists.

In here, a feasible solution means a vector \( x \) in \( \mathbb{R}^n \) satisfying all the constraints in the problem.

Proof: Sufficiency. If (6) has no feasible solution, then the set
\[ \left\{ x^T Pf_j(x) \geq 0, x^T x - 1 = 0, j = 1, 2, \ldots, k \right\} \]
is empty. Thus, for any \( x_0 \in \mathbb{R}^n \), \( x_0 \neq 0 \), an integer \( q \in \{1, 2, \ldots, k\} \), \( q \neq i \) must exist such that \( \frac{x_0^T Pf_q}{\|x_0\|} < 0 \). Since \( x^T Pf_q(x) \) is generalized homogeneous, we claim that \( \frac{x_0^T Pf_q(x_0)}{\|x_0\|} < 0 \) must hold, which means that \( \{ x^T Pf_j, j = 1, 2, \ldots, k \} \) is strictly complete, and thus quadratic stability follows from Lemma 3.1. In fact, if \( x_0^T Pf_q(x_0) \geq 0 \), there must exist a convex combination of \( \frac{x_0}{\|x_0\|} \) and \( x_0 \), say,
\[ y = \alpha x_0 + (1 - \alpha) \frac{x_0}{\|x_0\|} = (\alpha + 1 - \alpha) x_0 \]
such that \( y^T Pf_q(y) = 0 \), which contradicts Lemma 3.2.

If (6) has a feasible solution and the optimal value is negative, then for any \( x \in \mathbb{R}^n \), \( x \neq 0 \),
\[ \frac{x^T}{\|x\|} Pf(j) \leq 0 \]
holds as long as \( \frac{x^T}{\|x\|} Pf(j) \geq 0 \), \( j = 1, 2, \ldots, k \), and \( j \neq i \).

Necessity. Again from the generalized homogeneity of \( x^T Pf(x) \) we have \( x^T Pf(x) < 0 \), which means that \( \{ x^T Pf_j, j = 1, 2, \ldots, k \} \) is strictly complete.

Sufficiency. Suppose system (1) is quadratically stable. Lemma 3.1 says that \( \{ x^T Pf_j, j = 1, 2, \ldots, k \} \) is strictly complete. If (6) has a feasible solution \( x_0 \), that is,
\[ x_0^T Pf_j(x_0) \geq 0, \quad j \neq i, \]
\[ x_0^T x_0 - 1 = 0, \]
it follows immediately \( x_0^T Pf(x_0) < 0 \) from the strict completeness of \( \{ x^T Pf_j, j = 1, 2, \ldots, k \} \), and thus the optimal value is negative.

Now, we give the main new result of this contribution on the quadratic stability of switched systems.

**Theorem 3.2.** Suppose \( x^T Pf_j(x) \) is generalized homogeneous, \( j = 1, 2, \ldots, k \). System (1) is quadratically stable with respect to the quadratic Lyapunov function \( x^T P x \) if and only if an integer \( q \in \{1, 2, \ldots, k\} \) exists, which make the following two conditions to hold:

**Condition 1.**
\[ H_i - x^T Pf_j x - (H^T H)^{-1} (H^T H_i - H^T x^T Pf_j x) = 0, \quad x^T Pf_j = 0, \quad j = j_1, j_2, \ldots, j_q, \]
\[ x^T Pf_j \geq 0, \quad j \neq i, \]
\[ x^T x - 1 = 0 \]
(8)

has no solution or any solution \( x \) satisfies \( x^T Pf(x) < 0 \).

**Condition 2.**
\[ \sum_{j \neq i} \lambda_j(x) H_j = 0, \]
\[ x^T Pf j \geq 0, \quad j \neq i, \]
\[ x^T x - 1 = 0 \]
(9)

has no solution or any solution \( x \) satisfies \( x^T Pf(x) < 0 \), where
\[ \lambda_j(x) = 0, \quad \sum_{j \neq i} \lambda_j(x) = 1, \]
\[ \lambda_j(x) x^T Pf j = 0, \quad j \neq i. \]

**Proof.** Necessity. Assume that System (1) is quadratically stable. From Lemma 5 we know \( \{ x^T Pf_j, j = 1, 2, \ldots, k \} \) is strictly complete. If (8) or (9) has solution \( x \), then, \( x^T Pf(x) \geq 0, j \neq i \). Therefore \( x^T Pf(x) < 0 \) follows from the strict completeness of \( \{ x^T Pf_j, j = 1, 2, \ldots, k \} \).

Sufficiency. Suppose that (8) and (9) have no solutions or have solution \( x \) satisfying \( x^T Pf(x) < 0 \). If (6) has no feasible solution, then a global
optimal solution must exist because all feasible solutions are limited to the sphere $x^T x = 1$. According to Theorem 3.1, we need only to show that the optimal value is negative.

Let $x$ be an optimal solution to (6). From Fritz John condition there exist a set of constants $\lambda_i$, $i = 1, 2, \ldots, k$, not all zero, and a constant $\mu$ satisfying

$$\bar{x}_i (Pf_i + \frac{\partial f_i}{\partial x} P x) + \sum_{j \neq i} \lambda_j (Pf_j + \frac{\partial f_j}{\partial x} P x) + \mu x = 0,$$

$$\bar{x}_j x^T Pf_j = 0, j \neq i,$$

$$\bar{x}_j \geq 0, j = 1, 2, \ldots, k. \quad (10)$$

Multiplying (10) by $x^T$ from left and taking into account $x^T x = 1$, we have

$$\bar{x}_i (x^T Pf_i + x^T \frac{\partial f_i}{\partial x} P x) + \sum_{j \neq i} \bar{x}_j x^T \frac{\partial f_j}{\partial x} P x + \mu x = 0 \quad (11)$$

Now, we split the proof into two cases.

**Case 1** $\bar{\lambda}_i \neq 0$.

Denote $\lambda_j = \bar{x}_j / \bar{\lambda}_i$, $j \neq i$ and $\mu = \bar{\mu} / \bar{\lambda}_i$. Equations (10) become

$$(Pf_i + \frac{\partial f_i}{\partial x} P x) + \sum_{j \neq i} \lambda_j (Pf_j + \frac{\partial f_j}{\partial x} P x) + \mu x = 0,$$

$$\lambda_j x^T Pf_j = 0, j \neq i,$$

$$\lambda_j \geq 0, j = 1, 2, \ldots, k,$$

and Eq. (11) gives

$$\mu = -(x^T Pf_i + x^T \frac{\partial f_i}{\partial x} P x) - \sum_{j \neq i} \lambda_j x^T \frac{\partial f_j}{\partial x} P x \quad (12)$$

Substituting this into (12) does result in

$$H_i - x^T Pf_i x + \sum_{j \neq i} \lambda_j H_j = 0 \quad (14)$$

Note that $\lambda_j = 0$ if $j \notin M_i(x)$, and for any $j \in Q_i(x)$, $\exists \alpha_{j_m}(x)$, $m \in S_i(x)$ such that

$$H_j(x) = \sum_{m \in S_i(x)} \alpha_{j_m}(x) H_m(x).$$

Therefore (14) gives

$$H_i - x^T Pf_i x + \sum_{j \neq i} \lambda_j \alpha_{j_m}(x) H_m(x).$$

where $\delta_m = \lambda_m + \sum_{j \neq i} \lambda_j \alpha_{j_m}(x)$. Now, by rewriting (15) in matrix-vector form we have

$$H_i - x^T Pf_i x + H_j \begin{pmatrix} \delta_{j_1} \\ \delta_{j_2} \end{pmatrix} = 0 \cdot (16)$$

Multiplying (16) by $H^T$ from left yields

$$H^T H_i - H^T x^T Pf_i x + H^T H \begin{pmatrix} \delta_{j_1} \\ \delta_{j_2} \end{pmatrix} = 0 \quad (17)$$

and therefore

$$\begin{pmatrix} \delta_{j_1} \\ \delta_{j_2} \end{pmatrix} = -(H^T H)^{-1} (H^T H_i - H^T x^T Pf_i x) \quad (18)$$

By substituting (18) into (16), we arrive at

$$H_i - x^T Pf_i x - H (H^T H)^{-1} (H^T H_i - H^T x^T Pf_i x) = 0 \cdot (19)$$

which is exactly the first equation of (8). Thus, $x^T Pf_i < 0$ follows from Condition 1.

**Case 2** $\bar{\lambda}_i = 0$.

Note that $\bar{x}_j$ are not all zero, we can denote

$$\lambda_j = \frac{\bar{x}_j}{\sum_{m \neq i} \bar{x}_m}, \mu = \frac{\bar{\mu}}{\sum_{m \neq i} \bar{x}_m}$$

Rewriting the first equation of (10) as

$$\sum_{j \neq i} \lambda_j (Pf_j + \frac{\partial f_j}{\partial x} P x) + \mu x = 0 \quad (20)$$

and multiplying(20) by $x^T$ from left, we obtain
\[
\mu = - \sum_{j \neq i} \lambda_j \left( x^T P_f + \frac{\partial f_j}{\partial x}^T P x \right) .
\] (21)

Substituting (21) into (20) and noticing that \( \lambda_j x^T P f_j = 0 \), we arrive at
\[
\sum_{j \neq i} \lambda_j \left( P_f + \frac{\partial f_j}{\partial x}^T P x - x^T \frac{\partial f_j}{\partial x}^T P x \right) = 0
\] (22)
which is exactly the first equation of Condition 2. Thus, \( x^T P f_j < 0 \) indeed follows from Condition 2.

**Remark 3.2.** If \( f_j(x) \) is homogeneous, the condition \( x^T P f_j(x) \) being generalized homogeneous is automatically satisfied.

**Corollary 3.1.** If system (1) is a switched linear system, that is, \( f_j(x) = A_j x \), then system (1) is quadratically stable if and only if a positive definite matrix \( P \) and an integer \( i \) exist, which make the following two conditions to hold:

**Condition 1.**
\[
B_j x - H (H^T H)^{-1} H^T B_i x - x^T B_i x = 0,
\]
\[
x^T B_j x = 0, j = j_1, j_2, \ldots, j_k,
\]
\[
x^T B_j x \geq 0, j \neq i,
\]
\[
x^T x - 1 = 0
\]
has no solution or any solution \( x \) satisfies \( x^T B_j x < 0 \), where \( B_j = A_j^T P + P A_j, H = (b_{j_1}, b_{j_2}, \ldots, b_{j_k}) \).

**Condition 2.**
\[
\sum_{j \neq i} \lambda_j (x) B_j x = 0
\]
\[
x^T B_j x \geq 0, j \neq i,
\]
\[
x^T x - 1 = 0
\]
has no solution or any solution \( x \) satisfies \( x^T B_j x < 0 \), where \( \lambda_j(x) \geq 0, \sum_{j \neq i} \lambda_j(x) = 1, \lambda_j(x)x^T B_j x = 0, j \neq i \).

Yet as another corollary, we can easily show the well-known convex combination condition [11].

**Corollary 3.2.** If system (1) is a switched linear system with \( f_j(x) = A_j x \), and if in addition, there exists a convex combination \( A = \sum_i k \beta_i A_i, \sum_i k \beta_i = 1, \beta_i > 0 \), which is stable, then system (1) is quadratically stable.

### 4 Conclusion

Quadratic stability of switched non-linear systems has been studied via an alternative approach in this paper, and two new theorems proved. The key point is to avoid employing the strict completeness, which is usually very difficult to verify, and thus facilitate the stability analysis.

Instead, we first transform the problem of quadratic stability of systems into an equivalent restricted nonlinear programming problem. And, then we derive and prove a necessary and sufficient condition for quadratic stability by making use of Fritz John condition. The condition derived in here is much easier to verify in practical applications because only algebraic equations and inequalities are involved.

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### References


